OPTIMIZATION OF SYSTEMS WITH SENSITIVITY CONSTRAINTS

Ramon Ronald Owens

United States Naval Postgraduate School

THESIS

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by

Ramon Ronald Owens

Thesis Advisors: D. E. Kirk
H. A. Titus

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Sensitivity Constraints

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Ramon Ronald Owens
Lieutenant Commander, United States Navy
B.S., United States Naval Academy, 1958

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ABSTRACT

A cohesive and detailed treatment of the theory and engineering implications of trajectory sensitivity is presented. Fundamental results that provide insight into the theoretical aspects of trajectory sensitivity analysis, in both the frequency and time domains, are reviewed. Several related methods for incorporating sensitivity considerations in the design of systems are presented and used to solve a meaningful fifth-order numerical example: a flexible space vehicle in booster powered flight. Comparisons are made between an optimal design and designs which include a sensitivity term in the performance measure and conclusions are drawn about the efficacy of these techniques in control system design.

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I. INTRODUCTION

During recent years, the search for solutions to optimal control system problems has been quite intensive. In particular, optimal feedback control of linear dynamical systems has been studied extensively. This attention to feedback control is well placed. As Bode pointed out [1], feedback control is particularly desirable in that it 1) tends to stablize systems, 2) reduces the effects on the output due to extraneous noise or non-linear distortion in the plant, and 3) tends to reduce the effects on the system transfer characteristics caused by variation in plant parameters. The research in the area of optimal linear feedback control has resulted in a considerable body of knowledge about the theoretical aspects of the unconstrained linear state regulator problem. The necessary and sufficient conditions for the solution of this problem are well known [2, 3, 4]. Despite apparently complete knowledge concerning optimal linear state regulator control theory, its implementation in practical applications generally requires the use of digital computers and may be limited in many instances by cost considerations.

There are several other reasons why the optimal linear state regulator solution may not be implemented. These reasons are related to necessary conditions or assumptions required for the theoretical solution of the problem (all states available for feedback, control unconstrained and

time-varying) or engineering considerations concerned with formulation of the system model and its controller. Among the latter are considerations for the sensitivity of the response of a dynamic system to variations in plant parameters.

Every control problem, modern or classical, begins with the formulation of a system model. Generally, this model takes the form of one or more nonlinear time varying differential equations. This nontrivial process of system modelling, perhaps after much refinement, can ultimately result in a system described by a set of first-order ordinary differential state equations [5] of the form

$$\underline{\dot{x}}(t) = \underline{a}[\underline{x}(t),\underline{q}(t),\underline{u}(t),t] \tag{1.1}$$

where $\underline{x}(t)$ is the state n-vector

$\underline{q}(t)$ is the variable parameter r-vector

$\underline{u}(t)$ is the control input m-vector

t is the independent variable, usually time.

A physical system mathematically modelled by equations of the form of (1.1) are the subject of extensive design and analysis techniques. The results of these techniques are generally assumed to be valid and applicable to the physical system. The validity of this assumption requires careful scrutiny.

No matter how careful and precise the engineer is in establishing his model, there will always exist differences between the behavior of the physical system and that predicted by equations like (1.1). The vector function, $\underline{a}(.)$,

in (1.1) cannot, in general, be exactly determined. The parameters $\underline{q}(t)$ are, for many meaningful systems, difficult to measure accurately and in any event will have tolerances associated with them. The solution of (1.1) will generally yield only to the approximate techniques of the digital or analog computer. Even if the modelling difficulties mentioned above did not exist, the problem of changes in system components due to aging, environmental changes and repair part exchanges would introduce modelling inaccuracies. It is clear that even under the very best of circumstances, differences will exist between the system and its model.

In order to obtain the advantages of linear optimal control theory, a nonlinear system may be solved for its optimal trajectory and then an approximate linear system may be formulated relating small variations from the nonlinear optimal trajectory. The approximations involved lead to model inaccuracies and again differences between system behavior and modelled behavior will exist.

In many physical plants, all of the states are not available for feedback. In this case, implementation of the optimal linear state regulator problem may not be feasible [3]. However, if the system is completely observable, the states can be computed from the output [4]. Schemes for obtaining the system states from the system output include the use of observers [6] and Kalman filters [7]. However, these techniques use the system model in

their realizations and consequently introduce additional variations into the system.

Assuming that the modelling process and the model's solution is an accurate representation of the plant, translating the solution into controller signals or action is subject to many of the variations discussed above. Hence, implementation of the controller may introduce system inaccuracies.

Ignoring all of these difficulties, the "perfectly" modelled plant and "perfectly" implemented controller may still be inadequate. This "perfect" system may be unable to cope successfully with random or even deterministic disturbance inputs to which the system may be subjected.

Over the years as classical feedback theory has been developed, in recognition of the many uncertainties inherent in system design, a compatible theory for sensitivity analysis has also been developed [8]. This theory for the most part is based on Bode's definition [1] of sensitivity. From it developed a "Folk Theorem" [9] which states in effect that the sensitivity of the transfer characteristics of a system to parameter variations is reduced as the feedback gains are increased. This theorem works in many cases, and is frequently the only tool applied to sensitivity reduction in the design of many classical controllers.

Presently considerable research effort is being directed at the problem of developing sensitivity analysis

and reduced sensitivity design techniques compatible with modern control theory. In multiple-input multiple-output systems cast as optimal control problems interest has been concentrated in several system characteristics in addition to the system transfer function. These include the sensitivities of state trajectories, performance measures, system eigenvalues, and final states to variations of initial conditions and plant or controller parameters.

In this thesis, an historical review of sensitivity theory compatible with modern control theory is presented. Next a general theory for trajectory sensitivity analysis is reviewed and applied to the optimal state regulator problem. Then several current techniques proposed for designing optimal linear regulators with reduced sensitivities are investigated. Finally some of these techniques are applied to the solution of a flexible Saturn booster problem and the resulting numerical solutions are compared.

II. HISTORICAL REVIEW

A. CLASSICAL SENSITIVITY

The basic concepts that provide the foundation for modern sensitivity theory appeared in the fundamental work of Bode [1] which also marks the beginning of the modern theory of feedback systems. Bode defined feedback in terms of the return difference and at the same time established several important relationships between feedback and sensitivity. In the development of automatic control theory that followed, analysis and design formulations should have included sensitivity as an important adjunct. Such, however, was not the case. Until the beginning of the last decade, little can be found in the literature relating control systems and sensitivity. In an important exception [10], Truxal briefly discussed return difference and sensitivity within the context of signal flow graphs.

Horowitz [11] argued that the use of feedback loops to reduce sensitivity of the system transfer function to substantial plant parameter variations or random disturbances is as effective as the use of adaptive systems without their added complexity. He concluded, by using the classical sensitivity analysis techniques, that the inadequacies attributed to feedback systems by adaptive system designers were unfounded. Indeed, given the same design constraints, he demonstrated that whatever performance had been claimed for adaptive systems could be matched with considerably less complexity by feedback control systems. In

his book, Synthesis of Feedback Systems [12], Horowitz provided an excellent and complete treatment of classical control system design with sensitivity considerations. He applied the classical design techniques (root locus, Bode plots, Nyquist criterion) to the design of systems with reduced sensitivity to plant parameter variations and random input disturbances.

It should be noted that almost all of these early studies [1, 10, 11, 12] were confined to Laplace transform and frequency domain analysis of sensitivity. These techniques were applied mainly to determining the sensitivity of the system transfer function to changes of a system parameter. Root locus techniques were applied also to determine the sensitivity of pole-zero locations to variations of a system parameter [12]. Horowitz pointed out [12] that the desired system performance is generally a time domain specification, and that any correlation that exists between system frequency response and time response is approximate. If system time response is accurately required, then a time domain synthesis technique must be used.

Miller and Murray [13], in establishing a mathematically sound basis for error analysis for the solutions of ordinary differential equations by machine methods, derived the differential equation that describes the sensitivity coefficients of the system. This same "sensitivity equation" provided the time domain sensitivity analysis techniques described by Tomovic [8].

Kokotovic [14] proposed his "sensitivity points" method for linear systems by which the sensitivity of the system scalar output to variations of each of the plant parameters could be obtained. In fact, when implemented on the analog computer all of these sensitivities could be obtained simultaneously. This time domain method was extended to provide an automatic analog computer parameter minimization of the difference between a norm of the output trajectory of an analog system and the output of a specified standard model.

In a recent paper [15], Wilkie and Perkins presented a method of finding the sensitivities of all of the states of a linear time-invariant system to each of the system parameters simultaneously. The method required modelling the n-th order linear system in the companion canonic state form, plus an additional n-th order sensitivity model and a single-input n-th order system model for each input in excess of 1. This led to the requirement of 2m-1 n-th order models independent of the number of parameters for a system having m inputs. This was a considerable reduction in the number of models required over previous techniques, where an n-th order sensitivity model was required for each parameter considered. The Perkins-Wilkie method assumed that the linear system could be transformed into the companion canonic state form.

B. OPTIMAL CONTROL SENSITIVITY

It is interesting that the first published results concerning the sensitivity of optimal control systems [16] turned out to be a restatement of Bode's theorem relating return difference and sensitivity. Kalman, in solving the inverse problem of optimal control theory for linear state regulator problems, established a necessary condition for its solution. He showed that if a feedback control law is optimal, then the magnitude of the return difference for the feedback system must be greater than one. Additional results are obtained that tend to bridge the gap, to some extent, between classical and optimal control theory. Kalman's very complete discussion included important results concerning stability, improvement of sensitivity, and frequency-domain criteria for this essentially time domain problem.

In a development similar to that of [16], Perkins and Cruz [17] developed a sensitivity matrix operator which relates the open-loop and closed-loop system output variations. They established a sensitivity index which is a measure of the weighted mean square output error. By applying Parseval's theorem to the sensitivity index, they derived a frequency domain sufficiency condition that ensures sensitivity reduction. Additionally Perkins and Cruz demonstrated that if a system is designed to satisfy the sufficiency condition mentioned above, then the system is optimal in the sense that the constant, linear control law obtained minimizes an infinite-time quadratic performance

measure and at the same time transfers the states of a linear time-invariant completely controllable system from some initial state to the origin asymptotically.

Kreindler in [18] derived results similar to those of [16] and [17]. He established sufficient conditions which guarantee for multiple-input multiple-output linear regulator systems that the integral of a certain quadratic form of the closed-loop sensitivity is less than the corresponding integral for the open-loop system. Additionally, for single-input systems in the companion canonic form, the sufficiency condition ensured that the integral of the square of <u>each</u> closed-loop sensitivity trajectory will be less than the corresponding open-loop integral. This was a valuable result.

The analysis techniques discussed in the literature cited above have not resulted in established design techniques. Several authors [19, 20, 21, 22] defined a sensitivity differential equation after Miller and Murray [13] for each variable parameter, augmented the resulting differential equations to the system state equations, defined a performance measure with added quadratic terms measuring sensitivity, and then solved the resulting system using the Hamiltonian and Pontryagin's maximum principle to obtain a reduced sensitivity design. The resulting feedback control law was a linear combination of the system states and the augmented state sensitivities.

Dompe and Dorf [21] demonstrated with a second-order example, that Kahne's [19] design scheme does indeed result

in a feedback control system with reduced sensitivity. The control law did not result in the optimal solution of the mathematical model defined because Kahne neglected the effects of parameter variation on the feedback states.

Cassidy and Lee [22] included the terms excluded by Kahne.

D'Angelo, et.al., [20] proposed a reduced sensitivity design technique with an infinite-time performance measure resulting in constant feedback gains.

Higginbotham [23] included an additional term in the sensitivity differential equation that was neglected both by Kahne [19] and Cassidy and Lee [22]. He presented a comparison of [19] and [22] using a first-order example. By use of the example, he demonstrated that the Cassidy and Lee technique results in a lower "cost" than Kahne's technique. No comparison was made of the resulting sensitivity reduction, if any.

All of the above optimal sensitivity analysis and design techniques were concerned with transfer function or trajectory sensitivity. Considerable research has also been directed at the so-called performance sensitivity problem. The concern here is with variations in the performance index resulting from parameter variations. In a basic result, Pagurek [24] demonstrated that the performance index sensitivity to plant parameter variations for open-loop and closed-loop optimal linear state regulators are equal. This is an astounding result that seemed to imply the benefits of feedback with respect to plant parameter

variations were not available to optimal linear systems. This result was extended to a large class of nonlinear optimal control problems by Witsenhausen [25].

Although the results of [24] and [25] hold for infinitesimal parameter variations, Sinha and Atluri [26] showed that for finite but small variations the closed-loop performance index sensitivity is less than the open-loop sensitivity. Kreindler [18] made a similar argument in favor of feedback systems.

In a design procedure for minimizing performance measure sensitivity Rohrer and Sobral [27] introduced a "relative sensitivity" function. The procedure resulted in a minimax solution that is applicable for large parameter variations.

Salmon [28] generalized the Rohrer and Sobral procedure and proposed a new minimax algorithm for its solution.

Very recently Cassidy and Roy [29] described what appears to be a promising scheme for designing insensitive linear output regulator systems. By modifying the usual quadratic performance measure they were able to constrain the feedback coefficients multiplying the unmeasured states to zero. Thus output feedback was obtained. They defined the state sensitivity differential equations, adjoined them to the state equations to form an augmented state vector. A sensitivity term was added to the performance measure and their Specific Optimal Control (SOC) approach resulted in a constant feedback controller that did not require knowledge of the sensitivity variables nor the unmeasured states.

Much work in the area of sensitivity design remains to be done. All of the proposed design techniques have severe limitations that in general do not admit to practical application. These limitations and some proposals for improved techniques will be discussed subsequently.

III. SENSITIVITY ANALYSIS

The objective of sensitivity analysis is to quantitatively predict the effect of disturbances on the dynamic behavior of a system. This implies the development of a model of system sensitivity perhaps not unlike the system model. In fact, this is the case. When the system model is described by transform techniques, for example, generally the sensitivity analysis is developed using the same techniques [10, 11, 12]. Dynamical systems modelled in the time domain are frequently analyzed for sensitivity by transform operators [16, 17, 18] or in the time domain [13, 14, 15, 19, 20, 22, 23, 29]. The remainder of this thesis will be devoted to dynamical systems that can be described by simultaneous ordinary differential state equations of the form of (1.1).

A. NOTATION

The behavior of the plant is completely specified by the differential system

$$\begin{aligned} \dot{\underline{x}}(t) &= \underline{a}[\underline{x}(t), \underline{u}(t), \underline{q}(t), t] \\ \underline{x}(t_0) &= \underline{c} \end{aligned} \tag{3.1}$$

where $\underline{x}(t)$ is the real n-dimensional state vector, $\underline{u}(t)$ is the real n-dimensional control vector, $\underline{q}(t)$ is the real r-dimensional parameter vector, and $\underline{a}(.)$ is an n-dimensional vector function. In general, capital Roman letters will denote matrices, lower-case Roman letters vectors, and

lower case Greek letters scalars. Exceptions may occur in order to conform with general practice. These exceptions will be obvious.

B. TRANSFORM ANALYSIS

1. Single-Input Systems

The completely controllable [3, 30] constant linear dynamical system to be considered is described by

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{B}\underline{u}(t) \tag{3.2}$$

$$\underline{x}(t_0) = \underline{c}$$

which is optimized by the control law

$$\underline{u}(t) = \underline{F}\underline{x}(t) \tag{3.3}$$

with respect to the performance measure

$$J = \tfrac{1}{2} \int_0^\infty (\underline{x}'\underline{Q}\underline{x} + \underline{u}'\underline{R}\underline{u})dt \tag{3.4}$$

where $\underline{Q}$ is a non-negative definite constant symmetric matrix and $\underline{R}$ is a positive definite constant symmetric matrix. The prime denotes the transpose.

The solution to this problem in the form of a constant linear feedback control law (3.3) has been demonstrated by Kalman [3]. The matrix $\underline{F}$ is given by

$$\underline{F} = -\underline{R}^{-1}\underline{B}'\underline{K} \tag{3.5}$$

where $\underline{K}$ is the positive definite symmetric solution of the Riccati matrix equation

$$\underline{K}\underline{B}\underline{R}^{-1}\underline{B}'\underline{K} - \underline{K}\underline{A} - \underline{A}'\underline{K} - \underline{Q} = \underline{0}. \tag{3.6}$$

Kalman [16] considered such a system restricted to a single input $\mu(t)$, described by

$$\dot{\underline{x}}(t) = \underline{\underline{A}}\underline{x}(t) + \underline{b}\mu(t) \tag{3.7}$$

$$\underline{x}(0) = \underline{c} \ .$$

Figure 1 is a block diagram of the structure of this system. This system can be represented in Laplace transform notation, ignoring initial conditions, by

$$s\underline{X}(s) = \underline{\underline{A}}\underline{X}(s) + \underline{b}U(s). \tag{3.8}$$

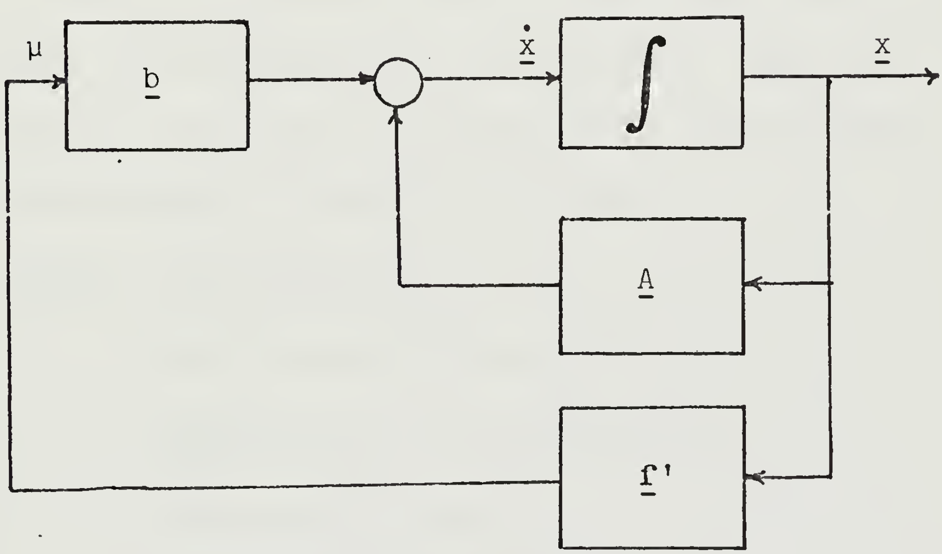

Figure 1. Linear single-input optimal control system.

From (3.8) one can obtain

$$\underline{X}(s) = \underline{\Phi}(s)\underline{b}U(s), \tag{3.9}$$

where $\underline{\Phi}(s) = (s\underline{I}-\underline{A})^{-1}$ is the Laplace transform of the fundamental matrix. Multiplying both sides of (3.9) by $\underline{f}'$, yields the scalar equation

$$\underline{f}'\underline{X}(s) = \underline{f}' \ \underline{\Phi}(s)\underline{b}U(s).$$

With $N(s) = \underline{f}'\underline{X}(s)$, then

$$N(s)/U(s) = \underline{f}'\underline{\Phi}(s)\underline{b} \tag{3.10}$$

is the open-loop transfer function.

The homogeneous, closed-loop form equation for the system of Figure 1 is given by

$$\dot{\underline{x}} = (\underline{A}+\underline{b}\underline{f}')\underline{x}. \tag{3.11}$$

The fundamental matrix for system (3.11) is

$$\underline{\Phi}_k(s) = (s\underline{I}-\underline{A}-\underline{b}\underline{f}')^{-1}.$$

In the following development the open-loop and closed-loop characteristic polynomials will be required. They are $\Psi(s) = \det(s\underline{I}-\underline{A})$ and $\Psi_k(s) = \det(s\underline{I}-\underline{A}-\underline{b}\underline{f}')$ respectively.

Continuing the algebra as follows

$$\begin{aligned}
\Psi_k(s) &= \det(s\underline{I}-\underline{A}-\underline{b}\underline{f}') \\
&= \det[\, s\underline{I}-\underline{A}-(s\underline{I}-\underline{A})(s\underline{I}-\underline{A})^{-1}\underline{b}\underline{f}'] \\
&= \det[(s\underline{I}-\underline{A})\ (\underline{I}-(s\underline{I}-\underline{A})^{-1}\underline{b}\underline{f}')] \\
&= \det[(s\underline{I}-\underline{A})(\underline{I}-\underline{\Phi}(s)\underline{b}\underline{f}')] \\
&= \det\ (s\underline{I}-\underline{A})\det(\underline{I}-\underline{\Phi}(s)\underline{b}\underline{f}') \\
&= \Psi(s)\det(\underline{I}-\underline{\Phi}(s)\underline{b}\underline{f}') \\
\Psi_k(s) &= \Psi(s)(1-\underline{f}'\underline{\Phi}(s)\underline{b})
\end{aligned}$$

finally yields

$$\Psi_k(s)/\Psi(s) = (1-\underline{f}'\underline{\Phi}(s)\underline{b}), \tag{3.12}$$

where the last step depends on a determinantal identity proved in [31]. The quantity $(1-\underline{f}'\underline{\Phi}(s)\underline{b})$ is the classical

return difference. Solving (3.12) for $\underline{f}'\underline{\Phi}(s)\underline{b}$ and substituting into (3.10) yields

$$N(s)/U(s) = \frac{\Psi(s) - \Psi_k(s)}{\Psi(s)} \tag{3.13}$$

Since $\underline{Q}$ is non-negative definite and symmetric there is a matrix $\underline{P}$ such that $\underline{P}'\underline{P} = \underline{Q}$. Using this fact and letting the term $\underline{u}'\underline{R}\underline{u} = \mu^2$ (single-input system), equations (3.4), (3.5), and (3.6) can be written as:

$$J = \tfrac{1}{2} \int_0^\infty [(\underline{x}'\underline{P}'\underline{P}\underline{x}) + \mu^2]dt \tag{3.14}$$

$$\underline{f}' = -\underline{b}'\underline{K} \tag{3.15}$$

and

$$\underline{K}\underline{b}\underline{b}'\underline{K} - \underline{K}\underline{A} - \underline{A}'\underline{K} - \underline{P}'\underline{P} = \underline{0}. \tag{3.16}$$

Kalman [16] proves that the $\underline{K}$ which satisfies (3.15) and (3.16) also satisfies

$$-\underline{f}\underline{f}' - \underline{K}(\underline{A}+\underline{b}\underline{f}') - (\underline{A}'+\underline{f}\underline{b}')\underline{K} - \underline{P}'\underline{P} = \underline{0} \tag{3.17}$$

and that:

Theorem 1. Given a completely controllable constant linear single-input system (3.7) and the performance measure (3.14) such that the pair $[\underline{A},\underline{P}]$ is completely observable [3, 30], a necessary and sufficient condition for $\underline{f}$ to be a constant stable optimal control law is that there exist a matrix $\underline{K}$ which satisfies the algebraic relations

$$\underline{K} = \underline{K}' \text{ is positive definite} \tag{3.18}$$

$$\underline{f} = -\underline{K}\underline{b} \tag{3.19}$$

$$-\underline{K}(\underline{A}+\underline{b}\underline{f}') - (\underline{A}'+\underline{f}\underline{b}')\underline{K} = \underline{P}'\underline{P}+\underline{f}\underline{f}'. \tag{3.17}$$

Equation (3.17) together with $\underline{K} = \underline{K}'$ is positive definite implies $\underline{f}$ is a stable control law and that $(\underline{A} + \underline{b}\underline{f}')$ is a stability matrix according to Lyapunov stability theory [5].

Theorem 1 provides a relationship that connects $\underline{f}$ and $\underline{P}$. However the relationship is not easily interpreted and is certainly not very useful as a design tool. Kalman continues his development and finds another relationship between $\underline{f}$ and $\underline{P}$ in which $\underline{K}$ has been eliminated.

Adding and subtracting $s\underline{K}$ in equation (3.16) gives

$$\underline{K}(s\underline{I}-\underline{A}) + (-s\underline{I}-\underline{A}')\underline{K} = \underline{P}'\underline{P} - \underline{K}\underline{b}\underline{b}'\underline{K},$$

and

$$\underline{K}[\underline{\Phi}(s)]^{-1} + [\underline{\Phi}'(-s)]^{-1}\underline{K} = \underline{P}'\underline{P} - \underline{K}\underline{b}\underline{b}'\underline{K}. \qquad (3.20)$$

Premultiplying and post multiplying (3.20) by $\underline{b}'\underline{\Phi}'(-s)$ and $\underline{\Phi}(s)\underline{b}$ respectively yields the scalar equation

$$\underline{b}'\underline{\Phi}'(-s)\underline{K}\underline{b} + \underline{b}'\underline{K}\underline{\Phi}(s)\underline{b} = \underline{b}'\underline{\Phi}'(-s)[\underline{P}'\underline{P} - \underline{K}\underline{b}\underline{b}'\underline{K}]\underline{\Phi}(s)\underline{b}.$$

Substituting $\underline{f} = -\underline{K}\underline{b}$ yields

$$-\underline{b}'\underline{\Phi}'(-s)\underline{f} - \underline{f}'\underline{\Phi}(s)\underline{b} = \underline{b}'\underline{\Phi}'(-s)\underline{P}'\underline{P}\underline{\Phi}(s)\underline{b} - \underline{b}'\underline{\Phi}'(-s)\underline{f}\underline{f}'\underline{\Phi}(s)\underline{b}.$$

which can be written

$$[1 - \underline{b}'\underline{\Phi}'(-s)\underline{f}]\,[1-\underline{f}'\underline{\Phi}(s)\underline{b}] = 1 + \underline{b}'\underline{\Phi}'(-s)\underline{P}'\underline{P}\underline{\Phi}(s)\underline{b}.$$

Substituting $s = j\omega$ gives the desired result.

$$|1 - \underline{f}'\underline{\phi}(j\omega)|^2 = 1 + ||\underline{P}\underline{\phi}(j\omega)\underline{b}||^2 \qquad (3.21)$$

where the notation $||\underline{Z}||^2$ implies the quadratic form $\underline{Z}'\underline{Z}$.

Kalman's result [16] can now be stated:

Theorem 2. Given a completely controllable constant linear single-input system (3.7) and the performance

measure (3.14) such that the pair $[\underline{A},\underline{P}]$ is completely observable, a necessary and sufficient condition for $\underline{f}$ to be a constant optimal control law is that $\underline{f}$ be stable and that (3.21) hold for all ω.

Equation (3.21) implies that for $\underline{f}$ to be a constant optimal control law, the magnitude of the return difference must be greater than unity for all ω, i.e.,

$$|T(j\omega)| = |1 - \underline{f}'\underline{\phi}(j\omega)\underline{b}|>1. \tag{3.22}$$

This result ensures that stable control laws satisfying (3.22) will provide a feedback system with reduced sensitivities to variations of parameters in the plant compared to the same plant with an equivalent open-loop control. It also implies that the larger the return difference, the less sensitive the plant will be. This is shown by demanding that the return difference $T_1(j\omega)$ for control law $\underline{f}_1$ be greater in magnitude than $T_2(j\omega)$ for $\underline{f}_2$. That is

$$|T_1(j\omega)/T_2(j\omega)|>1 \tag{3.23}$$

will ensure that the system with control law $\underline{f}_1$ will be less sensitive than the same system with control law $\underline{f}_2$.
This follows from

$$\begin{aligned} |T_1(j\omega)| &= |\psi_1(j\omega)/\psi(j\omega)| \\ |T_2(j\omega)| &= |\psi_2(j\omega)/\psi(j\omega)| \end{aligned} \tag{3.24}$$

and equation (3.12).

Although this formulation provides a means of evaluating various control laws, it does not appear to be very useful as a design procedure. If a means of finding

the matrix $\underline{Q}$ which satisfies the conditions of theorem 2 can be obtained, a design procedure of practical utility could be established. It seems that further investigation of this aspect of the problem might be fruitful.

Kalman's result guarantees that any constant optimal stable control law for a completely controllable constant single-input plant which minimizes the performance measure (3.14) subject to the constraints of (3.7) will also satisfy (3.21). Therefore, such a control law satisfies (3.22) and the resulting closed-loop system will have reduced sensitivity when compared to the same system with an equivalent open-loop control. This is an important result and adds considerable meaning to the term "optimal" for this problem.

Approaching the problem of constructing a $\underline{Q}$ matrix that results in an optimal control law $\underline{f}$ in the sense that equation (3.22) is satisfied, from (3.12) and (3.22) one can write

$$|1 - \underline{f}'\underline{\Phi}(s)\underline{b}|^2 = |\Psi_k(s)/\Psi(s)|^2 > 1 \tag{3.25}$$

and therefore, (3.26)

$$[1 - \underline{f}'(-s\underline{I}-\underline{A})^{-1}\underline{b}]\ [1 - \underline{f}'(s\underline{I}-\underline{A})^{-1}\underline{b}] = \frac{\Psi_k(-s)\Psi_k(s)}{\Psi(-s)\Psi(s)} > 1,$$

or

$$\frac{\Psi_k(-s)\Psi_k(s) - \Psi(-s)\Psi(s)}{\Psi(-s)\Psi(s)} > 0. \tag{3.27}$$

In the above equations, the relation

$$|\Psi(s)|^2 = \Psi(s)\Psi(-s) \tag{3.28}$$

which holds for polynomials with real coefficients has been used. In order to evaluate (3.27), using the fact that the pair $[\underline{A},\underline{b}]$ is completely controllable, the system

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{b}\mu$$

can be transformed into the companion form by $\underline{z} = \underline{T}\underline{x}$. Proceeding as follows

$$\underline{T}\dot{\underline{x}} = \underline{T}\underline{A}\underline{T}^{-1}\underline{T}\underline{x} + \underline{T}\underline{b}\mu$$

or

$$\dot{\underline{z}} = \underline{C}\underline{z} + \underline{d}\mu$$

where

$$\underline{C} = \begin{Bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & & & & . \\ 0 & 0 & 0 & & & & . \\ . & & & . & & & . \\ . & & & & . & & . \\ . & & & & & . & 1 \\ -\alpha_1 & -\alpha_2 & . & . & . & & -\alpha_n \end{Bmatrix}$$

and

$$\underline{d} = \begin{Bmatrix} 0 \\ . \\ . \\ . \\ 1 \end{Bmatrix} .$$

Since the characteristic polynomial is invariant under the linear, non-singular transformation $\underline{T}$, $\Psi(s)$ can be written as

$$\Psi(s) = \det(s\underline{I} - \underline{A}) = \det(s\underline{I} - \underline{C})$$

and similarly for $\Psi_k(s)$. The control law for this system is

$$\mu = \underline{f}'\underline{z}$$

where

$$\underline{f}' = \{ \gamma_1 \ \gamma_2 \ \gamma_3 \ \cdots \ \gamma_n \} .$$

A well known method for finding the inverse is expressed by

$$\underline{M}^{-1} = \frac{\text{adj}\underline{M}}{\det\underline{M}} .$$

Using this technique yields for $(s\underline{I} - \underline{C})^{-1}\underline{d}$

$$(s\underline{I} - \underline{C})^{-1}\underline{d} = \frac{[\text{adj}(s\underline{I} - \underline{C})]\underline{d}}{\det(s\underline{I} - \underline{C})}$$

$$= \frac{1}{\Psi(s)}\left\{\begin{matrix} 1 \\ s \\ s^2 \\ \cdot \\ \cdot \\ \cdot \\ s^{n-1} \end{matrix}\right\} .$$

Premultiplying by f',

$$\underline{f}'(s\underline{I} - \underline{C})^{-1}\underline{d} = \frac{\gamma_n s^{n-1} + \gamma_{n-1}s^{n-2} + \cdots + \gamma_1}{s^n + \alpha_n s^{n-1} + \alpha_{n-1}s^{n-2} + \cdots + \alpha_1} . \quad (3.29)$$

From the definition of $\underline{d}$ and $\underline{f}$ one can obtain

$$\underline{d}\underline{f}' = \begin{Bmatrix} 0 & 0 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \gamma_1 & \gamma_2 & \cdots & \gamma_n \end{Bmatrix}$$

and

$$\underline{C} + \underline{d}\underline{f}' = \begin{Bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & \cdots & 1 \\ \gamma_1 - \alpha_1 & \gamma_2 - \alpha_2 & \cdots & \cdots & \gamma_n - \alpha_n \end{Bmatrix}$$

Since $\Psi_k(s) = \det(s\underline{I} - \underline{C} - \underline{d}\underline{f}')$, it follows that

$$\underline{f}'(s\underline{I} - \underline{C})^{-1}\underline{d} = \frac{\Psi(s) - \Psi_k(s)}{\Psi(s)} \quad . \tag{3.30}$$

This same result is determined directly from (3.12), but in the present development $\Psi(s) - \Psi_k(s)$ is determined from (3.29) and (3.30) to be

$$\Psi(s) - \Psi_k(s) = \gamma_n s^{n-1} + \gamma_{n-1} s^{n-2} + \ . \ . \ . + \gamma_1 . \tag{3.31}$$

Inequality (3.26) can be rewritten as

$$\frac{\Psi_k(s)\Psi_k(-s) - \Psi(s)\Psi(-s)}{\Psi(s)\Psi(-s)} > 0 . \tag{3.32}$$

The numerator polynomial of (3.32) can be factored as

$$\Psi_k(s)\Psi_k(-s) - \Psi(s)\Psi(-s) = \delta(s)\delta(-s) \tag{3.33}$$

where $\delta(s)$ is a polynomial, of degree at most n-1, having all its zeros in the left-half s-plane. Defining the Hurwitz polynomial

$$\delta(s) = g_n s^{n-1} + g_{n-1} s^{n-2} + \ldots + g_1 \quad (3.34)$$

and the vector

$$\underline{g} = \begin{Bmatrix} g_1 \\ g_2 \\ \cdot \\ \cdot \\ \cdot \\ g_n \end{Bmatrix}, \quad (3.35)$$

the inequalities (3.26) and (3.32) can be combined with (3.33) to yield

$$[1 - \underline{f}'(-s\underline{I} - \underline{C})^{-1}\underline{d}]\ [1 - \underline{f}'(s\underline{I} - \underline{C})^{-1}\underline{d}] = \frac{\delta(s)\delta(-s)}{\Psi(s)\Psi(-s)} + 1 \quad (3.36)$$

Defining

$$\underline{g}'(s\underline{I} - \underline{C})^{-1}\underline{d} = \frac{1}{\Psi(s)}\ [g_n s^{n-1} + g_{n-1} s^{n-2} + \ldots + g_1]\ , \quad (3.37)$$

analogous to (3.29) and substituting (3.37) into equation (3.36) yields

$$\begin{aligned}[1 - \underline{f}'(-s\underline{I} - \underline{C})^{-1}\underline{d}]\ [1 - \underline{f}'(s\underline{I} - \underline{C})^{-1}\underline{d}]& \\ = [\underline{d}'(-s\underline{I} - \underline{C}^{-1})\underline{g}\underline{g}'(s\underline{I} - \underline{C})\underline{d}] + 1.& \quad (3.38)\end{aligned}$$

Kalman has shown that a $\underline{Q}$ constructed from

$$\underline{Q} = \underline{g}\underline{g}'$$

will yield the stable control law, $\underline{f}'$, which satisfies theorem 2 and is indeed optimal. That is, the pair $[\underline{A},\underline{g}]$ is completely observable.

In the above development, since the characteristic polynomial is invariant under transformation $\underline{T}$, $\underline{A}$ and $\underline{b}$ can be substituted for $\underline{C}$ and $\underline{d}$ respectively in equation (3.38).

Is the development useful as a design technique? Clearly it is not since constructing the proper $\underline{Q}$ depends on factoring $\Psi_k(s)\Psi_k(-s) - \Psi(s)\Psi(-s)$. Knowledge of $\Psi_k(s)$ requires knowledge of $\underline{f}'$, the vector the design is supposed to yield. Hence this development has little practical utility, but it does provide considerable insight into the complexities of the problem.

2. Multiple-Input Systems

The following discussion for multiple-input multiple-output systems presents sufficient conditions which ensure that a feedback system will have reduced output sensitivity compared to an equivalent open-loop system. In one development Perkins and Cruz [17] determine the sufficiency condition for finite output errors. In a similar development, Kreindler [18] proves that the same sufficient condition holds for an output sensitivity function. Because the derivations are quite similar, they are developed simultaneously after appropriate definitions are presented.

The linear time-invariant system considered is described by the state-variable differential equation.

$$\dot{\underline{x}} = \underline{A}(\underline{q})\underline{x} + \underline{B}(\underline{q})\underline{u} \;. \tag{3.39}$$

The plant input, $\underline{u}$, can be expressed as an open-loop control or a closed-loop control law that is assumed to be linear in $\underline{x}$. In order that the following comparison be meaningful, it

is assumed that initial conditions and external inputs, $\underline{r}$, to the system are the same for both the open and closed-loop systems. Additionally, it is assumed that any variations in the plant parameters, $\underline{q}$, are the same for both systems. The objective here is to ensure that any variations in the system output result only from equivalent parameter variations. Open-loop and closed-loop systems that satisfy these assumptions are considered <u>equivalent</u> systems. The equivalent closed and open-loop systems are shown in Figure 2 and Figure 3 respectively.

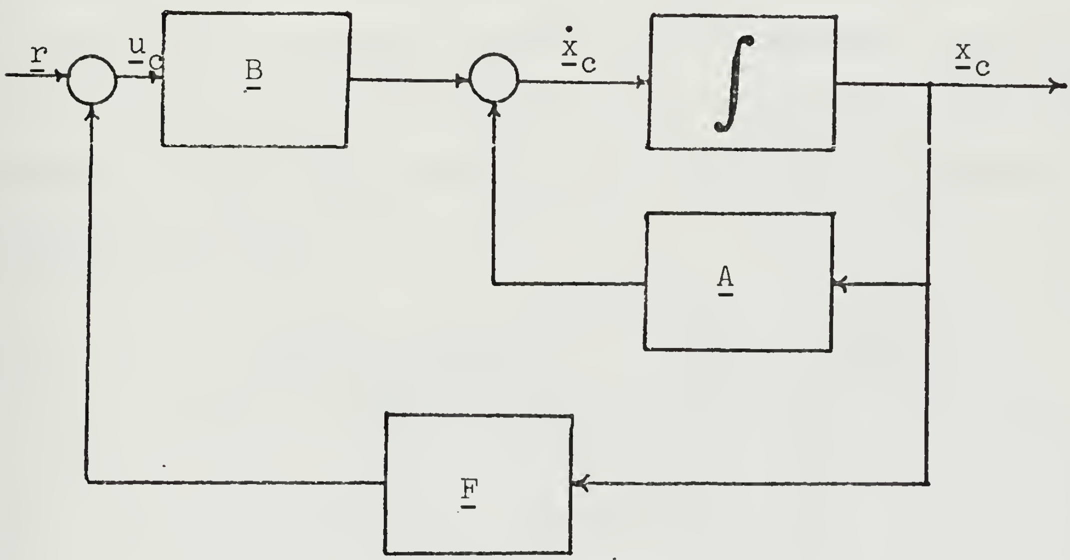

Figure 2. Closed-loop system configuration.

Considering the case where the vector $\underline{q}$ has one element q, the nominal parameter value is defined by $q = q_0$. The plant input-output transfer function is

$$\underline{T}(s,q) = (s\underline{I} - \underline{A})^{-1}\underline{B}. \tag{3.40}$$

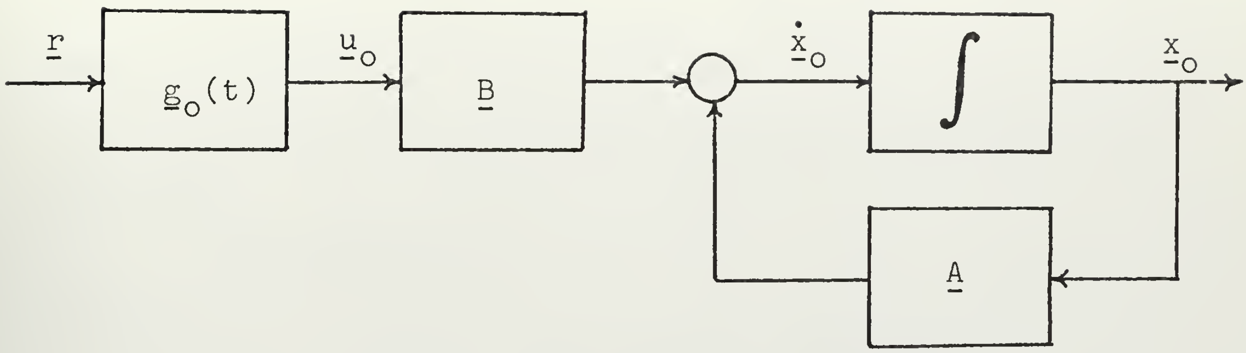

Figure 3. Open-loop system configuration.

This definition holds for the identical plants of each of the systems under consideration. Figure 2 and Figure 3 can be represented in block diagram form by Figure 4 and Figure 5 respectively. The closed-loop outputs and open-loop outputs are $\underline{X}_c$ and $\underline{X}_o$ respectively. The input notation is analogous to this.

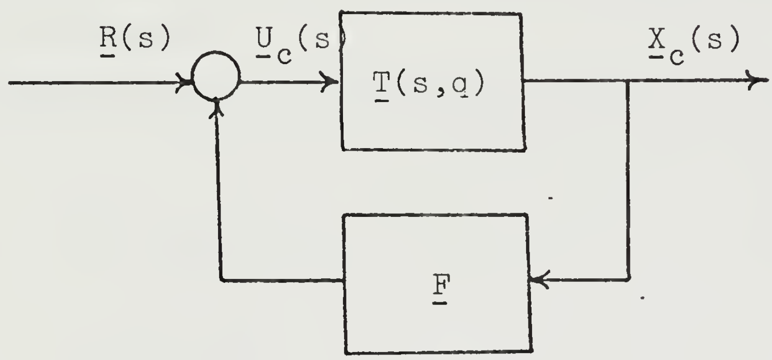

Figure 4. Closed-loop block diagram.

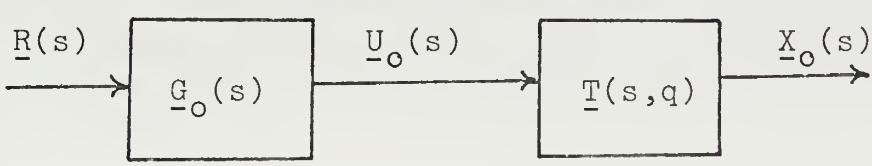

Figure 5. Open-loop block diagram.

The controller $\underline{G}_o$ is constructed such that for $\underline{T} = \underline{T}(s,q_o)$, $\underline{X}_c(s,q_o) = \underline{X}_o(s,q_o)$. The problem then is to study the effects of plant parameter variations on $\underline{X}_c(s,q)$ and $\underline{X}_o(s,q)$ under the assumption that the variations are identical.

From Figure 4

$$\underline{X}_c(s,q) = \underline{T}(s,q)[\underline{R}(s) + \underline{F}\underline{X}_c(s)]$$

or

$$\underline{X}_c(q) = [\underline{I} - \underline{T}(q)\underline{F}]^{-1}\underline{T}(q)\underline{R}. \tag{3.41}$$

From Figure 5

$$\underline{X}_o(s,q) = \underline{T}(s,q)\underline{G}_o(s)\underline{R}(s)$$

or

$$\underline{X}_o(q) = \underline{T}(q)\underline{G}_o\underline{R}. \tag{3.42}$$

In (3.41) and (3.42) and in the remainder of this development the functional dependence on s will not be written for notational convenience.

Assuming that the perturbed parameter q is given by

$$q = q_o + \Delta q, \tag{3.43}$$

then

$$\underline{T}(q) = \underline{T}(q_o + \Delta q). \tag{3.44}$$

Clearly the variation of q will cause $\underline{X}_c$ and $\underline{X}_o$ to be perturbed from their nominal values. Defining this perturbation as $\underline{E}_c$ and $\underline{E}_o$ for the closed and open-loop cases, respectively, then,

$$\underline{E}_c = \underline{X}_c(q_o) - \underline{X}_c(q_o + \Delta q) \tag{3.45}$$

$$\underline{E}_o = \underline{X}_o(q_o) - \underline{X}_o(q_o + \Delta q) \,. \tag{3.46}$$

Using (3.42) $\underline{E}_o$ can be written as

$$\underline{E}_o = [\underline{T}(q_o) - \underline{T}(q)]\underline{U}_o. \tag{3.47}$$

Recalling that $\underline{U}_c$ is not independent of q, one writes

$$\begin{aligned}\underline{U}_c(q) &= \underline{R} + \underline{F}\underline{X}_c(q)\\ &= \underline{R} + \underline{F}[\underline{X}_c(q) - \underline{X}_c(q_o)] + \underline{F}\underline{X}_c(q_o)\\ \underline{U}_c(q) &= \underline{U}_c(q_o) + \underline{F}[\underline{X}_c(q) - \underline{X}_c(q_o)]. \end{aligned} \tag{3.48}$$

Using equations (3.41), (3.45), and (3.48), the closed-loop output perturbation can be written as

$$\begin{aligned}\underline{E}_c &= \underline{T}(q_o)\underline{U}_c(q_o) - \underline{T}(q)\underline{U}_c(q)\\ &= \underline{T}(q_o)\underline{U}_c(q_o) - \underline{T}(q)\underline{U}_c(q_o)\\ &\quad - \underline{T}(q)\underline{F}[\underline{X}_c(q) - \underline{X}_c(q_o)]\\ &= [\underline{T}(q_o) - \underline{T}(q)]\underline{U}_c(q_o) + \underline{T}(q)\underline{F}\underline{E}_c\\ &= [\underline{I} - \underline{T}(q)\underline{F}]^{-1}[\underline{T}(q_o) - \underline{T}(q)]\underline{U}_c(q_o)\\ \underline{E}_c &= [\underline{I} - \underline{T}(q)\underline{F}]^{-1}\underline{E}_o \end{aligned} \tag{3.49}$$

where (3.47) has also been used.

In equation (3.49), $\underline{T}(q)$ depends on the parameter variation, however for differentially small plant-parameter variations (3.49) can be approximated by

$$\underline{E}_c = [\underline{I} - \underline{T}(q_o)\underline{F}]^{-1}\underline{E}_o \,. \tag{3.50}$$

This is the central result of [17]. Observing that $\underline{T}(q_o) = [s\underline{I} - \underline{A}(q_o)]^{-1}\underline{B}(q_o) = \underline{\Phi}(s,q_o)\underline{B}(q_o)$ and substituting into (3.50) yields

$$\underline{E}_c = [\underline{I} - \underline{\Phi}(s,q_o)\underline{B}(q_o)\underline{F}]^{-1}\underline{E}_o \ . \qquad (3.51)$$

The term $[\underline{I} - \underline{\Phi}(s,q_o)\underline{B}(q_o)\underline{F}]$ in (3.51) is similar to the scalar return difference and has an interpretation as the generalized matrix-return difference [17].

Kreindler [18], instead of defining an output error function, defines a sensitivity function

$$\underline{v}_c(t) = \partial\underline{x}_c(t)/\partial q = \lim_{\Delta q \to 0} \underline{e}_c(t)/\Delta q \qquad (3.52)$$

and

$$\underline{v}_o(t) = \partial\underline{x}_o(t)/\partial q = \lim_{\Delta q \to 0} \underline{e}_o(t)/\Delta q \qquad (3.53)$$

where the Laplace transforms of $\underline{v}_c(t)$ and $\underline{v}_o(t)$ are the vectors $\underline{V}_c(s)$ and $\underline{V}_o(s)$ respectively. With these definitions, the following relationship between $\underline{V}_c(s)$ and $\underline{V}_o(s)$ is obtained

$$\underline{V}_c(s) = [\underline{I} - \underline{\Phi}(s,q_o)\underline{B}(q_o)\underline{F}]^{-1}\underline{V}_o(s) \ . \qquad (3.54)$$

Before expressing the conditions under which the system sensitivity is reduced, it will be convenient to define a new sensitivity variable which under the proper conditions represents either $\underline{v}(t)$ or $\underline{e}(t)$. The new output sensitivity variable $\underline{y}(t)$ with Laplace transform $\underline{Y}(s)$ is defined such that

$$\underline{Y}_c(s) = [\underline{I} - \underline{\Phi}(s,q_o)\underline{B}(q_o)\underline{F}]^{-1}\underline{Y}_o(s) \qquad (3.55)$$

or

$$\underline{Y}_c(s) = \underline{S}(s,q_o)\underline{Y}_o(s) \ .$$

A measure of the output sensitivity is the weighted sum of the integrals of the sensitivities squared. Using this measure, the closed-loop system will have reduced sensitivity compared to the equivalent open-loop system if

$$\int_0^\infty \underline{y}'_c(t)\underline{Z}\underline{y}_c(t)dt < \int_0^\infty \underline{y}'_o(t)\underline{Z}\underline{y}_o(t)dt \ , \qquad (3.56)$$

for all t >0, where $\underline{Z}$ is a positive definite weighting matrix. Using Parseval's theorem, equation (3.56) implies

$$\int_{-\infty}^\infty \underline{Y}'_c(-j\omega)\underline{Z}\underline{Y}_c(j\omega)d\omega < \int_{-\infty}^\infty \underline{Y}'_o(-j\omega)\underline{Z}\underline{Y}_o(j\omega)d\omega$$

which is equivalent to

$$\int_{-\infty}^\infty \underline{Y}_o(-j\omega)[\underline{S}'(-j\omega)\underline{Z}\underline{S}(j\omega) - \underline{Z}]\ \underline{Y}_o(j\omega)d\omega<0. \qquad (3.57)$$

Inequality (3.57) will hold if

$$\underline{S}'(-j\omega)\underline{Z}\underline{S}(j\omega) - \underline{Z} <0 \qquad (3.58)$$

for all ω, where $\underline{S}(j\omega)$ is the inverse of the return difference. Premultiplying and postmultiplying (3.58) by $[\underline{S}'(-j\omega)]^{-1}$ and $[\underline{S}(j\omega)]^{-1}$ respectively yields

$$[\underline{I} - \underline{\phi}(-j\omega)\underline{B}\underline{F}]'\underline{Z}[\underline{I} - \underline{\phi}(j\omega)\underline{B}\underline{F}] - \underline{Z} > 0. \qquad (3.59)$$

The result is that any feedback control law $\underline{F}$ which satisfies either condition (3.57) or (3.59) will provide a feedback system with reduced sensitivity in the sense of (3.56). The integrals (3.56) can be integrated to obtain a measure of the amount of sensitivity reduction. Nevertheless, these conditions, (3.57) and (3.59), provide no hints about how to

design systems having reduced sensitivity. They might be useful in trial-and-error design but that is not very satisfying.

In a development similar to Kalman's [16], Perkins and Cruz demonstrate that the control law $\underline{F}$ that satisfies condition (3.58) is optimal for single-input linear time-invariant regulator systems with respect to the performance measure

$$J = \tfrac{1}{2}\int_0^\infty [\underline{x}'(t)\underline{\underline{Q}}\underline{x}(t) + \mu^2(t)]dt \quad . \qquad (3.60)$$

Kreindler [18] proves the following:

Theorem 3. For the completely controllable single-input linear plant in companion canonic form, for each component Y^i, $i = 1,2,\ldots,n$, of $\underline{Y}$, the following holds

$$Y_o^i(s) = [1 - \underline{f}'\underline{\Phi}(s)\underline{b}]Y_c^i(s). \qquad (3.61)$$

If in addition $\underline{f}$ and $\dot{\underline{x}} = \underline{\underline{A}}\underline{x} + \underline{b}\mu$ are optimal with respect to (3.60), then, for all $t_1>0$

$$\int_0^{t_1} [y_c^i(t)]^2 dt < \int_0^{t_1} [y_o^i(t)]^2 dt, \qquad (3.62)$$

$$i = 1, 2,\ldots,n.$$

This is a useful result. However as Kreindler [32] points out it is only applicable for systems in the companion canonic form.

The analysis techniques discussed above are exempt from a restriction that the analysis techniques discussed subsequently have almost universally: there was no identification of the parameter q made, hence, any and all of the

variable plant-parameters will exhibit reduced sensitivity when the conditions for reduced sensitivity are met.

C. STATE TRAJECTORY SENSITIVITY

In general, a plant which is sensitive to perturbations of its component values may be characterized in the form of equations (3.1). In the sensitivity analysis of such a system, it is necessary to relate numerically the dispersion of the solutions of (3.1) for varying values of the parameters, $\underline{q}$.

Although a great deal of work has been done recently in the area of state trajectory sensitivity, few important general results have been discovered. Most of the design techniques that are discussed in the literature result from application of the sensitivity equation [8, 13], obtained by taking the partial derivative of equation (3.1) as follows:

$$\frac{\partial}{\partial q_i}\left(\frac{d\underline{x}}{dt}\right) = \frac{\partial}{\partial q_i}\left[\ \underline{a}(\underline{x}(t),\underline{u}(t),\underline{q}(t),t)\right]$$

$$\frac{\partial}{\partial q_i}\left(\frac{d\underline{x}}{dt}\right) = \frac{\partial \underline{a}}{\partial \underline{x}} \cdot \frac{\partial \underline{x}}{\partial q_i} + \frac{\partial \underline{a}}{\partial \underline{u}} \cdot \frac{\partial \underline{u}}{\partial q_i} + \frac{\partial \underline{a}}{\partial \underline{q}} \cdot \frac{\partial \underline{q}}{\partial q_i} \ . \qquad (3.63)$$

The form of the matrix of partial derivatives, $\partial\underline{a}/\partial\underline{q}$, is defined by

$$\left(\frac{\partial \underline{a}}{\partial \underline{q}}\right)_{ij} = \frac{\partial a_i}{\partial q_j} \qquad \begin{array}{l} i = 1,2,\ldots n \\ j = 1,2,\ldots r. \end{array} \qquad (3.64)$$

The matrices $\partial\underline{a}/\partial\underline{x}$ and $\partial\underline{a}/\partial\underline{u}$ are dimensioned nXn and nXm respectively and are defined in a manner analogous to (3.64).

The form of the vector of partial derivatives, $\partial \underline{x}/\partial q_i$, is defined by

$$\frac{\partial \underline{x}}{\partial q_i} = \begin{Bmatrix} \frac{\partial x_1}{\partial q_i} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial x_n}{\partial q_i} \end{Bmatrix} . \tag{3.64a}$$

The vectors $\partial \underline{u}/\partial q_i$ and $\partial \underline{q}/\partial q_i$ are m and r-vectors respectively and are defined in a manner analogous to (3.64a).

If $\partial \underline{x}/\partial q_i$, $\partial \underline{x}/\partial t$, and $\partial \dot{\underline{x}}/\partial q_i$ are all continuous functions of $\underline{q}$ and t, the order of differentiation can be interchanged and

$$\frac{\partial}{\partial q_i} \left(\frac{d\underline{x}}{dt}\right) = \frac{d}{dt} \left(\frac{\partial \underline{x}}{\partial q_i}\right). \tag{3.65}$$

The sensitivity function (influence coefficient) $\underline{s}_i(t)$ is defined by

$$\underline{s}_i(t) = \frac{\partial \underline{x}(t)}{\partial q_i} \quad \text{for all } \underline{x} \text{ and } q_i, \tag{3.66}$$

$$i = 1,2,\ldots r.$$

Using equation (3.64) and definition (3.65), equation (3.63) can be written more compactly as

$$\dot{\underline{s}}_i = \frac{\partial \underline{a}}{\partial \underline{x}} \underline{s}_i + \frac{\partial \underline{a}}{\partial \underline{u}} \cdot \frac{\partial \underline{u}}{\partial q_i} + \frac{\partial \underline{a}}{\partial \underline{q}} \cdot \frac{\partial \underline{q}}{\partial q_i} \tag{3.67}$$

Since in section B it was shown that feedback control frequently can provide reduced sensitivity over the equivalent

open-loop system, it is assumed that a control law of the form

$$\underline{u}(t) = \underline{u}[\underline{x}(t)] \tag{3.68}$$

is specified. Under this assumption $\partial\underline{u}/\partial q_i$ can be written as:

$$\frac{\partial \underline{u}}{\partial q_i} = \frac{\partial \underline{u}}{\partial \underline{x}} \cdot \frac{\partial \underline{x}}{\partial q_i} = \frac{\partial \underline{u}}{\partial \underline{x}} \; \underline{s}_i . \tag{3.69}$$

Using (3.69), (3.67) can be written as

$$\dot{\underline{s}}_i = \left[\frac{\partial \underline{a}}{\partial \underline{x}} + \frac{\partial \underline{a}}{\partial \underline{u}} \cdot \frac{\partial \underline{u}}{\partial \underline{x}}\right] \underline{s}_i + \frac{\partial \underline{a}}{\partial \underline{q}} \cdot \frac{\partial \underline{q}}{\partial q_i}, \quad i = 1,..,r. \tag{3.70}$$

Rewriting (3.70), where the meaning of $\underline{A}_1(t)$ and $\underline{\omega}_i(t)$ is obvious, yields

$$\dot{\underline{s}}_i(t) = \underline{A}_1(t)\underline{s}_i(t) + \underline{\omega}_i(t), \quad i = 1,...r. \tag{3.71}$$

Equation (3.71) is the vector notation for nr equations each having the form

$$\frac{d}{dt}\left(\frac{\partial x_j}{\partial q_i}\right) = \sum_{k=1}^{n} \frac{\partial a_j}{\partial x_k} \cdot \frac{\partial x_k}{\partial q_i} + \sum_{k=1}^{n} \sum_{\ell=1}^{m} \frac{\partial a_j}{\partial u_\ell} \cdot \frac{\partial u_\ell}{\partial x_k}$$

$$\cdot \frac{\partial x_k}{\partial q_i} + \sum_{k=1}^{r} \frac{\partial a_j}{\partial q_k} \cdot \frac{\partial q_k}{\partial q_i} \tag{3.71a}$$

where $\partial q_k/\partial q_i = \begin{cases} 1 \text{ if } i = k \\ 0 \text{ otherwise.} \end{cases}$

In several equations above for convenience in notation, function arguments were omitted. This practice will prevail throughout this section.

Since it is assumed that the parameter variations have no effect on the initial conditions of (3.71),

$$\underline{s}_i(t_0) = \underline{0}. \tag{3.71b}$$

In order to obtain the sensitivity function defined by (3.71), the values for $\underline{A}_1(t)$ and $\underline{\omega}_i(t)$ must be determined. This amounts to evaluating the partial derivatives $\partial\underline{a}/\partial\underline{x}$, $\partial\underline{a}/\partial\underline{q}$, $\partial\underline{a}/\partial\underline{u}$, $\partial\underline{u}/\partial\underline{x}$, and $\partial\underline{q}/\partial q_i$. This can be done for the specific optimal control problem under consideration.

The optimal control is the admissible control, $\underline{u}^*(t)$, which causes the system

$$\dot{\underline{x}}(t) = \underline{a}[\underline{x}(t),\underline{u}(t),\underline{q}(t),t] \tag{3.1}$$

$$\underline{x}(t_0) = \underline{c}$$

to follow an admissible trajectory, $\underline{x}(t)$, that minimizes the cost functional

$$J = h[\underline{x}(t_f),t_f] + \int_{t_o}^{t_f} f[\underline{x}(t),\underline{u}(t)]dt \tag{3.72}$$

while transferring the state of the system from a given initial position $\underline{x}(t_0) = \underline{c}$ to some final position $\underline{x}(t_f) = \underline{x}_f$, that is restricted to some (n + 1) dimensional subset T of state-time space. T is called the target set. Here, the optimal control is constrained to the feedback form

$$\underline{u}^*(t) = \underline{u}[\underline{x}^*(t)], \tag{3.73}$$

where $\underline{u}^*$ and $\underline{x}^*$ are the optimal control and trajectory respectively.

Returning to consideration of the partial derivatives, $\partial \underline{a}/\partial \underline{x}$, $\partial \underline{a}/\partial \underline{u}$, and $\partial \underline{a}/\partial \underline{q}$ are all functions of $[\underline{x}(t), \underline{u}(t), \underline{q}(t), t]$. However, using (3.73), the explicit dependence on $\underline{u}(t)$ can be eliminated and then the partial derivatives become functions of $[\underline{x}(t), \underline{q}(t), t]$. The value $\underline{q}(t) = \hat{\underline{q}}(t)$ defines the point in parameter-time space at which the partial derivatives are to be evaluated. Additionally, it is noted that $\partial \underline{u}/\partial \underline{x}$ is also a function of $\underline{x}(t)$ and therefore, because of feedback, implicitly a function of $\underline{q}(t)$. Upon specification of $\hat{\underline{q}}(t)$ the differential equations (3.71) with their initial conditions (3.71b) can be solved for the sensitivity function, $\underline{s}_i(t)$. If small parameter variations are assumed the partial derivatives can be approximated by letting $\hat{\underline{q}} = \underline{q}_0$, where $\underline{q}_0$ represents the nominal value of $\underline{q}$.

It would be well at this point to summarize the development.

For the dynamical system

$$\dot{\underline{x}}(t) = \underline{a}[\underline{x}(t), \underline{u}(t), \underline{q}(t), t] \tag{3.1}$$

$$\underline{x}(t_0) = \underline{c} \ ,$$

the optimal control

$$\underline{u}^*(t) = \underline{u}[\underline{x}^*(t)] \tag{3.73}$$

minimizes

$$J = h[\underline{x}(t_f), t_f] + \int_{t_o}^{t_f} f[\underline{x}(t), \underline{u}(t)]dt. \tag{3.72}$$

The trajectory sensitivity with respect to the parameter q_i is

$$\underline{s}_i(t) = \frac{\partial \underline{x}(t)}{\partial q_i} \quad \text{for all } \underline{x} \text{ and } q_i,\ i = 1,2,\ldots,r \tag{3.66}$$

and is the solution of

$$\dot{\underline{s}}_i(t) = \underline{A}_1(t)\underline{s}_i(t) + \underline{\omega}_i(t), \tag{3.71}$$

where

$$\underline{A}_1(t) = \left[\frac{\partial \underline{a}}{\partial \underline{x}} + \frac{\partial \underline{a}}{\partial \underline{u}} \cdot \frac{\partial \underline{u}}{\partial \underline{x}} \right]_{\underline{q}=\underline{q}_0} \tag{3.74}$$

and

$$\underline{\omega}_i(t) = \left. \frac{\partial \underline{a}}{\partial \underline{q}} \cdot \frac{\partial \underline{q}}{\partial q_i} \right|_{\underline{q}=\underline{q}_0}. \tag{3.75}$$

Chan and Chuang in [33] obtain a similar result by expanding the right-hand side of (3.1) about $[\underline{x},\underline{q}_0,\underline{u}(\underline{x}),t]$ in a first order Taylor's expansion. They obtain

$$d\underline{z}/dt = \underline{A}_1(t)\underline{z} + \underline{B}(t)\underline{\Gamma}(t) \tag{3.76}$$

$$\underline{z}(0) = \underline{z}_0$$

where $z_i = x_i^* - x_{io}^*$ is the i^{th} component of the variation along the optimal trajectory, and $\gamma_j = q_j - q_{jo}$ is the variation of the parameter q_j about its nominal value. The matrix $\underline{A}_1(t)$ is defined as in (3.74), $\underline{B}(t)$ is

$$\underline{B}(t) = \partial \underline{a}/\partial \underline{q}, \tag{3.77}$$

and the partial derivatives in (3.76) are evaluated along the optimal trajectory.

Writing the i^{th} equation of (3.76) in expanded form yields

$$\frac{dz_j}{dt} = \sum_{k=1}^{n} \left[\frac{\partial a_j}{\partial x_k} + \sum_{\ell=1}^{m} \frac{\partial a_j}{\partial u_\ell} \cdot \frac{\partial u_\ell}{\partial x_k} \right] z_k$$

$$+ \sum_{k=1}^{r} \frac{\partial a_j}{\partial q_k} \gamma_k \, . \tag{3.78}$$

Dividing both sides of (3.78) by γ_i yields

$$\frac{d}{dt} \left[\frac{(x_j - x_{jo})}{\gamma_i} \right] = \sum_{k=1}^{n} \left[\frac{\partial a_j}{\partial x_k} + \sum_{\ell=1}^{m} \frac{\partial a_j}{\partial u_\ell} \cdot \frac{\partial u_\ell}{\partial x_k} \right]$$

$$\cdot \frac{(x_k - x_{ko})}{\gamma_i} + \sum_{k=1}^{r} \frac{\partial a_j}{\partial q_k} \cdot \frac{\gamma_x}{\gamma_i} \, . \tag{3.79}$$

In the limit as $\gamma_i \to 0$ equation (3.79) becomes

$$\frac{d}{dt} \left[\frac{\partial x_j}{\partial q_i} \right] = \sum_{k=1}^{n} \left[\frac{\partial a_j}{\partial x_k} + \sum_{\ell=1}^{m} \frac{\partial a_j}{\partial u_\ell} \cdot \frac{\partial u_\ell}{\partial x_k} \right] \frac{\partial x_k}{\partial q_i}$$

$$+ \sum_{k=1}^{r} \frac{\partial a_j}{\partial q_k} \cdot \frac{\partial q_k}{\partial q_i}$$

which is identical to equation (3.71a).

It is important to note that the sensitivity equation derived above is always linear; this is true even though the dynamic system (3.1) may be nonlinear; therefore, equation (3.71) is always a system of linear ordinary differential equations with constant or variable coefficients. It should also be noted that (3.71) is not valid for the case in which $\underline{q}$ can vary in such a manner as to change the order of the state equations [8]. In most applications the parameter vector, $\underline{q}(t)$, is assumed to be a slowly-varying time function that can be approximated by the constant parameter vector, $\underline{q}$.

1. Linear System State Trajectory Sensitivity

In this section the state trajectory analysis above will be applied to a linear state regulator system. The results of this development will be used extensively in following chapters.

The linear regulator system is described by the state equations

$$\underline{\dot{x}}(t) = \underline{A}(t,\underline{q})\underline{x}(t) + \underline{B}(t,\underline{q})\underline{u}(t) \tag{3.80}$$

with a linear feedback control law of the form

$$\underline{u}(t) = \underline{F}(t)\underline{x}(t) \; . \tag{3.81}$$

$\underline{A}(t)$ is the real, time-varying, nXn system matrix.

$\underline{B}(t)$ is the real, time-varying, nXm distribution matrix.

$\underline{F}(t)$ is the real, time-varying, mXn gain matrix.

$\underline{u}(t)$ is the control m-vector.

$\underline{x}(t)$ is the state n-vector.

$\underline{q}$ is the constant parameter r-vector.

Assuming that the solutions to (3.80) are analytically dependent on the parameters $\underline{q}$, the partial derivative of (3.80) with respect to parameter q_i is

$$\frac{d}{dt}\left(\frac{\partial \underline{x}}{\partial q_i}\right) = \frac{\partial \underline{A}}{\partial q_i}\underline{x} + \underline{A}\frac{\partial \underline{x}}{\partial q_i} + \frac{\partial \underline{B}}{\partial q_i}\underline{u} + \underline{B}\frac{\partial \underline{u}}{\partial q_i} \; . \tag{3.82}$$

Defining, as before,

$$\underline{s}_i = \frac{\partial \underline{x}}{\partial q_i} \; , \tag{3.83}$$

and using (3.81), the sensitivity equation (3.82) can be written more compactly as

$$\dot{\underline{s}}_i = \partial\underline{A}_i\underline{x} + \partial\underline{B}_i\underline{u} + \underline{A}\underline{s}_i + \underline{B}\,\frac{\partial\underline{u}}{\partial q_i}\,. \tag{3.84}$$

Taking the partial derivative of (3.81), where it is assumed that $\underline{F}$ is not a function of q_i, yields

$$\frac{\partial\underline{u}}{\partial q_i} = \underline{F}\underline{s}_i\,. \tag{3.85}$$

Using (3.85) another useful form of (3.82) is obtained,

$$\dot{\underline{s}}_i = [\partial\underline{A}_i + \partial\underline{B}_i\underline{F}]\underline{x} + [\underline{A} + \underline{B}\underline{F}]\underline{s}_i\,, \tag{3.86}$$

where the meaning of $\partial\underline{A}_i$ and $\partial\underline{B}_i$ is clear from (3.67) and (3.82).

D. PERFORMANCE SENSITIVITY

In a design procedure in which the ideal controller is optimal for a wide range of parameter variations, Rohrer and Sobral [27] define a new sensitivity function, the "relative sensitivity" of $\underline{u}(\underline{q},t)$ at parameter "operating point" $\underline{q}$. This relative sensitivity is expressed by

$$S^r(\underline{u},\underline{q}) = \frac{J(\underline{u},\underline{q}) - J^*(\underline{u}^*,\underline{q})}{J^*(\underline{u}^*,\underline{q})}\,, \tag{3.87}$$

where $\underline{u}^*(\underline{q},t)$ is the optimal control such that

$$J^*(\underline{u}^*,\underline{q}) = \min_{\underline{u}} J(\underline{u},\underline{q}), \tag{3.88}$$

subject to constraints

$$\dot{\underline{x}}(t) = \underline{a}[\underline{x}(t),\underline{u}(t),\underline{q},t]. \tag{3.1}$$

Among the advantages cited for the use of such a performance sensitivity is that it is always positive. Additionally when the parameters $\underline{q}$ are such that $\underline{u}=\underline{u}^*$, $S^r(\underline{u},\underline{q})=0$. This provides a measure against which system performance can be compared.

The design technique proposed consists of a minimaximization of the relative sensitivity with respect to $\underline{u}$ and $\underline{q}$. The procedure consists of assuming a design criteria which defines the plant sensitivity $SP(\underline{u})$. Two suggested plant sensitivities are

$$SP(\underline{u}) = \max_{\underline{q}} \left[S^r(\underline{u},\underline{q}) \right] \tag{3.89}$$

and

$$SP(\underline{u}) = \mathop{E}_{\underline{q}} \left[S^r(\underline{u},\underline{q}) \right] \tag{3.90}$$

where E $[\cdot]$ indicates expected value. The design procedure then consists of choosing $\hat{\underline{u}}$ such that

$$SP(\hat{\underline{u}}) = \min_{\underline{u}} \left[SP(\underline{u}) \right] \tag{3.91}$$

In a second-order example with a single variable parameter, Rohrer and Sobral demonstrate that the use of "relative sensitivity" in conjunction with (3.89) yields a system that has reduced sensitivity and remains close to the optimal over the entire range of parameter variations.

Salmon [28] presents a new algorithm for the global solution of a minimax problem. The algorithm converges to the global solution in the presence or absence of saddle

points. The class of optimization problems for which the algorithm applies includes those having quadratic performance measures with linear time-invariant state equations and a constant gain linear control law (the infinite interval problem described by Kalman [3]).

In two examples, Salmon applies the algorithm using the Rohrer and Sobral "relative sensitivity" and the "absolute sensitivity,"

$$S^a(\underline{u},\underline{q}) = J(\underline{u},\underline{q}) - J^*(\underline{u}^*,\underline{q}), \qquad (3.92)$$

in each.

In the first example, which is the same as that used by Rohrer and Sobral, the performance using "relative sensitivity" exceeded the optimal performance by less than 1.9 percent for all allowable values of q. The performance using "absolute sensitivity" exceeded the optimal performance by about 50 percent for q = 0, the minimum allowable value of q.

In the second example, the control objective is to maintain the spacing among a string of three electronically coupled vehicles. The five state variables are the three vehicle velocities, and the two spacings between the center and end vehicles. The performance measure is the infinite-time integral of a quadratic form involving vehicle spacing deviations and control. In this example with nine variable parameters, the "absolute sensitivity" technique provided performance closer to optimal than did the "relative sensitivity" technique. This was to be expected since it is more

important that the controller be close to ideal when the performance measure, $J(\underline{u},\underline{q})$, is large.

Another feature of design techniques using "absolute" or "relative" sensitivity is that the structure of the controller must be specified by the designer. Hence, the technique is amenable to solution of the de-sensitized output regulator problem.

The minimax design technique, whether using "relative sensitivity" or "absolute sensitivity" apparently has advantages of considerable value to the designer. However, there are also disadvantages. Minimax algorithms are difficult to implement except for restricted classes of problems. Additionally, the solutions obtained are dependent on the initial condition; hence, the solution is only valid for that initial condition. In general, the solution obtained by minimax design will be optimal for only one initial condition.

Özer [34] applies a minimax algorithm to the solution of the output regulator problem. He defines an auxiliary performance measure which takes the forms (3.87) and (3.92) among others. With a second-order example he demonstrates that the controllers obtained by the two methods differ widely. The problem is solved for several initial conditions and is used in an excellent discussion of the effects of initial conditions on performance for several auxiliary performance measures.

E. SUMMARY

In Kalman's development, it was shown that for the completely controllable single input plant

$$\dot{\underline{x}}(t) = \underline{\underline{A}}\underline{x}(t) + \underline{b}\mu(t) \tag{3.7}$$

and performance measure

$$J(\mu) = \tfrac{1}{2}\int_0^\infty [\underline{x}'\underline{\underline{Q}}\underline{x} + \mu^2]dt \tag{3.14}$$

the optimal control law

$$\mu^*(t) = \underline{f}'\underline{x}(t) \tag{3.93}$$

satisfies the condition

$$|1 - \underline{f}'\underline{\phi}(j\omega)\underline{b}|>1. \tag{3.22}$$

He also proved the inverse theorem: if a stable control law $\underline{f}$ satisfies (3.22), then it is optimal for a performance measure (3.14) with some $\underline{Q}$. Given the $\underline{f}$ one can find a $\underline{Q}$ by spectral factorization of a rational function.

An important result is that every optimal system of the form of (3.7), (3.14), and (3.93) will exhibit reduced sensitivity compared to an equivalent open-loop system.

Perkins and Cruz defined a sensitivity matrix

$$\underline{S}(s,\underline{q}) = [\underline{I} - \underline{\Phi}(s,\underline{q})\underline{B}(\underline{q})\underline{F}]^{-1}.$$

The inverse of this matrix was interpreted as the generalized matrix-return difference. Their central result was that constant feedback control laws, $\underline{F}$, for linear state regulator systems (3.39) that satisfy the condition

$$\underline{S}'(-j\omega)\underline{\underline{Z}}\underline{S}(j\omega) - \underline{Z} < 0 \tag{3.58}$$

for all ω, will provide reduced output error compared to the equivalent open-loop system in the sense that an integral inequality of the type (3.56) will hold, for $t \geq 0$.

Kreindler with parameters restricted to differentially small variations obtained the same result for the output sensitivity function (3.52) and (3.53). He also showed that for systems in the companion canonic form satisfying (3.58), the sensitivity measure (3.56) reduced to the form of (3.62). This result implies that the sensitivity of each state trajectory in the closed-loop system was less than the sensitivity of the corresponding open-loop state trajectory.

State trajectory sensitivity analysis was approached by means of the sensitivity function defined by Miller and Murray. A sensitivity differential equation was derived which was linear even though the system model could be nonlinear. The sensitivity differential equation was made explicit for linear state regulator systems as

$$\dot{\underline{s}}_i = \partial \underline{A}_i \underline{x} + \partial \underline{B}_i \underline{u} + \underline{\underline{A}}\underline{s}_i + \underline{B}\,\frac{\partial \underline{u}}{\partial q_i}\,. \tag{3.84}$$

Rohrer and Sobral introduced a new "relative sensitivity" performance measure that defines a minimax problem. The technique is applicable with large parameter variations. Salmon provided a minimax algorithm that could be used to solve problems using "relative sensitivity." However, by means of an example he demonstrated that for some problems, better results are obtained by using "absolute sensitivity"

for the performance measure than was obtained using "relative sensitivity." It was pointed out that the dependence on initial conditions severely limits the utility of these minimax schemes.

IV. TRAJECTORY SENSITIVITY DESIGN

A. GENERAL PROCEDURE

In the previous chapter, an expression for the trajectory sensitivity of a general feedback system to plant-parameter variations was developed. In this chapter that expression will be used in the development of an optimization problem that includes sensitivity constraints.

The objective is to use the trajectory sensitivity analysis technique previously established in formulating an optimization problem that will result in the design of controllers that are optimal in some sense and at the same time provide trajectory insensitivity to plant parameter disturbances.

Equations (3.66), (3.71), (3.74), and (3.75) are repeated here for convenience:

$$\underline{s}_i(t) = \frac{\partial \underline{x}(t)}{\partial\, q_i}\ , \quad i = 1,2,\ldots,r \tag{3.66}$$

$$\dot{\underline{s}}_i(t) = \underline{A}_1(t)\underline{s}_i(t) + \underline{\omega}_i(t) \tag{3.71}$$

$$\underline{s}_i(t_0) = \underline{0}$$

$$\underline{A}_1(t) = \left.\frac{\partial \underline{a}}{\partial \underline{x}} + \frac{\partial \underline{a}}{\partial \underline{u}} \cdot \frac{\partial \underline{u}}{\partial \underline{x}}\right|_{\underline{q}=\underline{q}_0} \tag{3.74}$$

$$\underline{\omega}_i(t) = \left.\frac{\partial \underline{a}}{\partial \underline{q}} \cdot \frac{\partial \underline{q}}{\partial q_i}\right|_{\underline{q}=\underline{q}_0} \tag{3.75}$$

By adjoining equation (3.71) to equation (3.1) the n(r+1) augmented state vector $\underline{z}$ is

$$\underline{z}(t) = \begin{Bmatrix} \underline{x}(t) \\ \underline{s}_1(t) \\ \cdot \\ \cdot \\ \cdot \\ \underline{s}_r(t) \end{Bmatrix} . \tag{4.1}$$

The corresponding initial condition vector is

$$\underline{z}(t_0) = \begin{Bmatrix} \underline{c} \\ \underline{0} \\ \cdot \\ \cdot \\ \cdot \\ \underline{0} \end{Bmatrix} . \tag{4.2}$$

The augmented system then is defined by

$$\dot{\underline{z}}(t) = \begin{Bmatrix} \underline{a}[\underline{x}(t),\underline{u}(t),\underline{q}(t),t] \\ \hline \underline{A}_1(t)\underline{s}_1(t) + \underline{\omega}_1(t) \\ \hline \cdot \\ \cdot \\ \cdot \\ \underline{A}_1(t)\underline{s}_r(t) + \underline{\omega}_r(t) \end{Bmatrix} \tag{4.3}$$

or more compactly as

$$\dot{\underline{z}}(t) = \underline{d}[\underline{x}(t),\underline{s}_1(t)\ldots\underline{s}_r(t),\underline{u}(t),\underline{q}(t),\underline{\omega}_1(t)\ldots\underline{\omega}_r(t),t]. \tag{4.4}$$

Following the procedure of Kahne [19] a new performance measure that includes a scalar sensitivity term

$$J = h[\underline{x}(t_f),t_f)]+\int_{t_o}^{t_f} [f(\underline{x}(t),\underline{u}(t)) + g(\underline{s}_1(t)\ldots\underline{s}_r(t))]dt \tag{4.5}$$

is defined.

Applying Pontryagin's minimum principle [2], the Hamiltonian is given by

$$H[\underline{z}(t),\underline{u}(t),\underline{p}(t),\underline{q}(t),t] = f(\cdot) + g(\cdot) + \underline{p}'\underline{d}(\cdot), \tag{4.6}$$

where $\underline{p}(t)$ are the Lagrange multipliers.

The necessary conditions for unconstrained $\underline{u}(t)$ to minimize the performance measure (4.5) are

$$\dot{\underline{z}}^*(t) = \frac{\partial H}{\partial \underline{p}} [\underline{z}^*(t),\underline{u}^*(t),\underline{p}^*(t),\underline{q}(t),t]$$

$$\dot{\underline{p}}^*(t) = -\frac{\partial H}{\partial \underline{z}} [\underline{z}^*(t),\underline{u}^*(t),\underline{p}^*(t),\underline{q}(t),t] \tag{4.7}$$

$$\underline{0} = \frac{\partial H}{\partial \underline{u}} [\underline{z}^*(t),\underline{u}^*(t),\underline{p}^*(t),\underline{q}(t),t]$$

for all $t_o \leq t \leq t_f$, where t_f is fixed and $\underline{x}(t_f)$ is free. The superscript (*) denotes optimal.

There is no guarantee that a solution to the above problem exists.

Considering the performance index (3.72), the controller obtained above will be suboptimal. A tradeoff between optimality with respect to (3.72) and sensitivity reduction will actually occur.

B. LINEAR REGULATOR SENSITIVITY DESIGN

In the preceeding section an optimization problem was defined and a method of solution outlined that resulted in

a control that was optimal in the sense that a performance measure which included sensitivity terms was minimized. There is some question concerning the existence of the solution proposed. In this section the procedure will be applied to the linear regulator problem. The solution to this problem exists and can be obtained following the formulation of Athans and Falb [4].

The linear time-varying state regulator system is defined by

$$\dot{\underline{x}}(t) = \underline{A}(t,\underline{q})\underline{x}(t) + \underline{B}(t,\underline{q})\underline{u}(t) \tag{4.8}$$

$$\underline{x}(t_o) = \underline{c}\ .$$

Rewriting equations (3.83) and (3.84) here for convenience

$$\underline{s}_i = \frac{\partial \underline{x}}{\partial q_i}\ ,\ i = 1,2\ldots r \tag{3.83}$$

$$\dot{\underline{s}}_i = \partial \underline{A}_i \underline{x} + \partial \underline{B}_i \underline{u} + \underline{A}\underline{s}_i + \underline{B}\,\frac{\partial \underline{u}}{\partial q_i} \tag{3.84}$$

$$\underline{s}_i(t_o) = \underline{0}.$$

The rn vector $\underline{s}$ is defined by

$$\underline{s} = \begin{Bmatrix} \underline{s}_1 \\ \cdot \\ \cdot \\ \cdot \\ \underline{s}_r \end{Bmatrix}\ . \tag{4.9}$$

The partial derivatives are evaluated at $\underline{q}_o$, the nominal value of $\underline{q}$. (Function arguments have been dropped for notational convenience.) Referring to (3.82) and (3.84), the matrices ∂A_i and ∂B_i are defined by

$$\partial \underline{A}_i = \frac{\partial \underline{A}}{\partial q_i} \tag{4.10}$$

and

$$\partial \underline{B}_i = \frac{\partial \underline{B}}{\partial q_i} \ . \tag{4.11}$$

Higginbotham [23] writes the partial derivative of $\underline{u} = \underline{u}(\underline{x}, \underline{s}_1, \underline{s}_2, \ldots \underline{s}_r, t)$ with respect to q_i as

$$\frac{\partial \underline{u}}{\partial q_i} = \frac{\partial \underline{u}}{\partial \underline{x}} \cdot \frac{\partial \underline{x}}{\partial q_i} + \sum_{j=1}^{r} \frac{\partial \underline{u}}{\partial \underline{s}_j} \cdot \frac{\partial \underline{s}_j}{\partial q_i} \ . \tag{4.12}$$

Substituting equation (4.12) into (3.84) yields

$$\dot{\underline{s}}_i = \partial \underline{A}_i \underline{x} + \left(\underline{A} + \underline{B} \frac{\partial \underline{u}}{\partial \underline{x}} \right) \underline{s}_i + \partial \underline{B}_i \underline{u} + \underline{B} \sum_{j=1}^{r} \frac{\partial \underline{u}}{\partial \underline{s}_j} \cdot \frac{\partial \underline{s}_j}{\partial q_i} \tag{4.13}$$

or

$$\dot{\underline{s}}_i = \partial \underline{A}_i \underline{x} + \hat{\underline{A}} \underline{s}_i + \partial \underline{B}_i \underline{u} + \underline{e}_i \ . \tag{4.14}$$

It has been assumed that the resulting controller will be a feedback controller of the states and sensitivity functions. This result will be demonstrated. It has been further assumed that the partials $\partial \underline{u}/\partial \underline{x}$ and $\partial \underline{u}/\partial \underline{s}_j$ are evaluated at $\underline{q}=\underline{q}_o$.

Proceeding as before, equations (4.14) are adjoined to equations (4.8) to form the augmented system

$$\begin{Bmatrix} \dot{\underline{x}} \\ \hline \dot{\underline{s}}_1 \\ \hline \dot{\underline{s}}_2 \\ \hline \vdots \\ \dot{\underline{s}}_r \end{Bmatrix} = \begin{Bmatrix} \underline{A} & \underline{0} & \underline{0} & \cdots & \underline{0} \\ \hline \partial \underline{A}_1 & \hat{\underline{A}} & \underline{0} & \cdots & \underline{0} \\ \hline \partial \underline{A}_2 & \underline{0} & \hat{\underline{A}} & \cdots & \underline{0} \\ \hline \vdots & \vdots & \vdots & & \vdots \\ \partial \underline{A}_r & \underline{0} & \underline{0} & \cdots & \hat{\underline{A}} \end{Bmatrix} \begin{Bmatrix} \underline{x} \\ \hline \underline{s}_1 \\ \hline \underline{s}_2 \\ \vdots \\ \underline{s}_r \end{Bmatrix} + \begin{Bmatrix} \underline{B} \\ \hline \partial \underline{B}_1 \\ \hline \partial \underline{B}_2 \\ \hline \vdots \\ \partial \underline{B}_r \end{Bmatrix} \underline{u} + \begin{Bmatrix} \underline{0} \\ \hline \underline{e}_1 \\ \hline \underline{e}_2 \\ \hline \vdots \\ \underline{e}_r \end{Bmatrix} \tag{4.15}$$

which can be written more compactly as

$$\dot{\underline{z}} = \underline{A}_1\underline{z} + \underline{B}_1\underline{u} + \underline{e} \tag{4.16}$$

$$\underline{z}(t_o) = \underline{z}_o = \begin{Bmatrix} \underline{c} \\ \underline{0} \\ \cdot \\ \cdot \\ \cdot \\ \underline{0} \end{Bmatrix}$$

where the meanings of $\underline{z}$, $\underline{A}_1$, $\underline{B}_1$, and $\underline{e}$ are clear from (4.15).

The quadratic performance measure, J, is

$$J = \tfrac{1}{2}\underline{x}'(t_f)\underline{D}\underline{x}(t_f) + \tfrac{1}{2}\underline{s}'(t_f)\underline{E}\underline{s}(t_f) + \tfrac{1}{2}\int_{t_o}^{t_f} [\underline{x}'(t)\underline{Q}(t)\underline{x}(t) + \underline{s}'(t)\underline{W}(t)\underline{s}(t) + \underline{u}'(t)\underline{R}(t)\underline{u}(t)]dt \tag{4.17}$$

where the terminal time t_f is specified and

$\underline{E}$ is a constant rnXnr positive semidefinite matrix

$\underline{D}$ is a constant nXn positive semidefinite matrix

$\underline{Q}(t)$ is an nXn positive semidefinite matrix

$\underline{W}(t)$ is an nrXnr positive semidefinite matrix

and $\underline{R}(t)$ is an mXm positive definite matrix.

The performance measure (4.17) can be rewritten in terms of $\underline{z}$ as

$$J = \tfrac{1}{2}\underline{z}'(t_f)\underline{D}_1\underline{z}(t_f) + \tfrac{1}{2}\int_{t_o}^{t_f} [\underline{z}'\underline{Q}_1\underline{z} + \underline{u}'\underline{R}\underline{u}]dt \tag{4.18}$$

where

$$\underline{D}_1 = \begin{Bmatrix} \underline{D} & \vdots & \underline{0} \\ \cdots & \vdots & \cdots \\ \underline{0} & \vdots & \underline{E} \end{Bmatrix}, \quad (r+1)nX(r+1)n \tag{4.19}$$

and

$$\underline{Q}_1 = \left\{\begin{array}{c:c} \underline{Q} & \underline{0} \\ \hdashline \underline{0} & \underline{W} \end{array}\right\}, \ (r+1)nX(r+1)n. \tag{4.20}$$

By applying Pontryagin's minimum principle, the necessary and sufficient conditions for the optimal control are obtained.

The Hamiltonian, H, for (4.16) and (4.18) is defined as follows

$$H = \tfrac{1}{2}\,[\underline{z}'Q_1\underline{z} + \underline{u}'\underline{R}\underline{u}] + \underline{p}'\ [\underline{A}_1\underline{z} + \underline{B}_1\underline{u} + \underline{e}] \tag{4.21}$$

where $\underline{p}(t)$, the (r+1)n costate vector, satisfies the differential equation

$$\dot{\underline{p}} = -\frac{\partial H}{\partial \underline{z}} \tag{4.22}$$

or

$$\dot{\underline{p}} = -\underline{Q}_1\underline{z} - \underline{A}_1'\underline{p}\ . \tag{4.23}$$

The unconstrained optimal control must satisfy the equation

$$\frac{\partial H}{\partial \underline{u}} = \underline{0} \tag{4.24}$$

or

$$\underline{R}\underline{u}^* + \underline{B}_1'\underline{p} = \underline{0}\ . \tag{4.25}$$

Solving (4.25) for the optimal control yields

$$\underline{u}^* = -\underline{R}^{-1}\underline{B}_1'\underline{p}\ . \tag{4.26}$$

In order to express $\underline{u}^*$ in the specified form of equation (3.81) substituting (4.26) into (4.16) yields

$$\dot{\underline{z}} = \underline{A}_1\underline{z} - \underline{B}_1\underline{R}^{-1}\underline{B}_1'\underline{p} + \underline{e} \tag{4.27}$$

which when combined with (4.23) forms the augmented system

$$\begin{Bmatrix} \dot{\underline{z}} \\ --- \\ \dot{\underline{p}} \end{Bmatrix} = \left\{\begin{array}{c|c} \underline{A}_1 & -\underline{B}_1 R^{-1} \underline{B}_1' \\ \hline -\underline{Q}_1 & -\underline{A}_1' \end{array}\right\} \begin{Bmatrix} \underline{z} \\ --- \\ \underline{p} \end{Bmatrix} + \begin{Bmatrix} \underline{e} \\ --- \\ \underline{0} \end{Bmatrix} . \tag{4.28}$$

This is a system of 2(r+1)n differential equations. In order to solve them, 2(r+1)n boundary conditions are required. The first (r+1)n boundary conditions are defined by equation (4.16)

$$\underline{z}(t_0) = \begin{Bmatrix} \underline{c} \\ --- \\ \underline{0} \end{Bmatrix} . \tag{4.29}$$

The remaining boundary conditions required are final conditions for the co-state equations. They are obtained from the transversality conditions (see Table 5-1 of [4]), with fixed terminal time, t_f, and $\underline{z}(t_f)$ free

$$\underline{p}(t_f) = \frac{\partial}{\partial \underline{z}} \left(\tfrac{1}{2} \underline{z}' \underline{D}_1 \underline{z} \right) \Bigg|_{\underline{z}=\underline{z}(t_f)} \tag{4.30}$$

or

$$\underline{p}(t_f) = \underline{D}_1 \underline{z}(t_f) . \tag{4.31}$$

The solution of (4.28) at time t_f can be written as

$$\begin{Bmatrix} \underline{z}(t_f) \\ ----- \\ \underline{p}(t_f) \end{Bmatrix} = \underline{\phi}(t_f,t) \begin{Bmatrix} \underline{z}(t) \\ ---- \\ \underline{p}(t) \end{Bmatrix} + \int_t^{t_f} \underline{\phi}(t,\tau) \begin{Bmatrix} \underline{e}(\tau) \\ ------ \\ \underline{0} \end{Bmatrix} d\tau , \tag{4.32}$$

where $\underline{\phi}(t_f,t)$ is the transition matrix for the system (4.28).

Partitioning $\underline{\phi}(t_f,t)$ into four (r+1)nX(r+1)n submatrices yields

$$\underline{\phi}(t_f,t) = \left\{\begin{array}{c|c} \underline{\phi}_{11}(t_f,t) & \underline{\phi}_{12}(t_f,t) \\ \hline \underline{\phi}_{21}(t_f,t) & \underline{\phi}_{22}(t_f,t) \end{array}\right\} . \tag{4.33}$$

Substituting (4.31) into (4.32) yields

$$\left\{\begin{array}{c} \underline{z}(t_f) \\ \hline \underline{D}_1\underline{z}(t_f) \end{array}\right\} = \left\{\begin{array}{c|c} \underline{\phi}_{11} & \underline{\phi}_{12} \\ \hline \underline{\phi}_{21} & \underline{\phi}_{22} \end{array}\right\} \left\{\begin{array}{c} \underline{z}(t) \\ \hline \underline{p}(t) \end{array}\right\} \tag{4.34}$$

$$+\int_t^{t_f} \left\{\begin{array}{c|c} \underline{\phi}_{11}(t,\tau) & \underline{\phi}_{12}(t,\tau) \\ \hline \underline{\phi}_{21}(t,\tau) & \underline{\phi}_{22}(t,\tau) \end{array}\right\} \left\{\begin{array}{c} \underline{e}(\tau) \\ \hline \underline{0} \end{array}\right\} d\tau .$$

Performing the indicated matrix multiplication yields

$$\left\{\begin{array}{c} \underline{z}(t_f) \\ \hline \underline{D}_1\underline{z}(t_f) \end{array}\right\} = \left\{\begin{array}{c} \underline{\phi}_{11}\underline{z} + \underline{\phi}_{12}\underline{p} \\ \hline \underline{\phi}_{21}\underline{z} + \underline{\phi}_{22}\underline{p} \end{array}\right\} + \int_t^{t_f} \left\{\begin{array}{c} \underline{\phi}_{11}(t,\tau)\underline{e}(\tau) \\ \hline \underline{\phi}_{21}(t,\tau)\underline{e}(\tau) \end{array}\right\} d\tau . \tag{4.35}$$

Performing additional algebra

$$\underline{D}_1[\underline{\phi}_{11}\underline{z} + \underline{\phi}_{12}\underline{p}\ ] + \int_t^{t_f} \underline{D}_1\underline{\phi}_{11}(t,\tau)\underline{e}(\tau)d\tau$$

$$= \underline{\phi}_{21}\underline{z} + \underline{\phi}_{22}\underline{p} + \int_t^{t_f} \underline{\phi}_{21}(t,\tau)\underline{e}(\tau)d\tau$$

$$[\underline{\phi}_{22} - \underline{D}_1\underline{\phi}_{12}]\underline{p} = [\underline{D}_1\underline{\phi}_{11} - \underline{\phi}_{21}]\underline{z}$$

$$+\int_t^{t_f} \underline{D}_1\underline{\phi}_{11}(t,\tau)\underline{e}(\tau)d\tau - \int_t^{t_f} \underline{\phi}_{21}(t,\tau)\underline{e}(\tau)d\tau$$

yields

$$\underline{p}(t) = [\underline{\phi}_{22}(t_f,t) - \underline{D}_1\underline{\phi}_{12}(t_f,t)]^{-1} \cdot$$

$$\left\{[\underline{D}_1\underline{\phi}_{11}(t_f,t)-\underline{\phi}_{21}(t_f,t)]\underline{z}(t)+\int_t^{t_f}[\underline{D}_1\underline{\phi}_{11}(t,\tau)-\underline{\phi}_{21}(t,\tau)]\underline{e}(\tau)d\tau\right\}. \qquad (4.36)$$

Equation (4.36) is the desired result; it relates $\underline{p}(t)$ and $\underline{z}(t)$, and with (4.26) yields the optimal feedback controller. Kalman [3] has proved the existence of the inverses in (4.36) for all t, $t_o \leq t \leq t_f$.

Writing (4.36) more compactly

$$\underline{p}(t) = \underline{K}(t)\underline{z}(t) + \underline{v}(t) \qquad (4.37)$$

and substituting into (4.26) yields the optimal control

$$\underline{u}^*(t) = -\underline{R}^{-1}(t)\underline{B}_1'(t)[\underline{K}(t)\underline{z}(t) + \underline{v}(t)] \qquad (4.38)$$

where

$$\underline{K}(t) = [\underline{\phi}_{22}(t_f,t) - \underline{D}_1\underline{\phi}_{12}(t_f,t)]^{-1}[\underline{D}_1\underline{\phi}_{11}(t_f,t)-\underline{\phi}_{21}(t_f,t)] \qquad (4.39)$$

and

$$v(t) = [\underline{\phi}_{22}(t_f,t) - \underline{D}_1\underline{\phi}_{12}(t_f,t)]^{-1} \cdot$$

$$\int_t^{t_f} [\underline{D}_1\underline{\phi}_{11}(t,\tau) - \underline{\phi}_{21}(t,\tau)]\underline{e}(\tau)d\tau. \qquad (4.40)$$

Since (4.39) and (4.40) are extremely difficult to evaluate, a simpler expression would be helpful. Again following [4] the time derivative of (4.37) yields

$$\dot{\underline{p}} = \dot{\underline{K}}\underline{z} + \underline{K}\dot{\underline{z}} + \dot{\underline{v}} \ . \qquad (4.41)$$

From (4.27)

$$\dot{\underline{z}} = \underline{A}_1\underline{z} - \underline{M}\underline{p} + \underline{e} \tag{4.42}$$

where

$$\underline{M} = \underline{B}_1\underline{R}^{-1}\underline{B}' \tag{4.43}$$

and from (4.23)

$$\dot{\underline{p}} = - \underline{Q}_1\underline{z} - \underline{A}_1'\underline{p}. \tag{4.23}$$

Substituting (4.37) into (4.41) yields

$$\dot{\underline{z}} = [\underline{A}_1 - \underline{M}\underline{K}]\ \underline{z} - \underline{M}\underline{v} + \underline{e}. \tag{4.44}$$

Substituting (4.43) into (4.40) yields

$$\dot{\underline{p}} = [\dot{\underline{K}} + \underline{K}\underline{A} - \underline{K}\underline{M}\underline{K}]\ \underline{z} - \underline{K}\underline{M}\underline{v} + \underline{K}\underline{e} + \dot{\underline{v}}. \tag{4.45}$$

Substituting (4.37) into (4.23) yields

$$\dot{\underline{p}} = [-\underline{Q}_1 - \underline{A}_1'\underline{K}]\ \underline{z} - \underline{A}_1'\underline{v}\ . \tag{4.46}$$

Subtracting (4.46) from (4.45) yields

$$[\dot{\underline{K}} + \underline{K}\underline{A}_1 - \underline{K}\underline{M}\underline{K} + \underline{A}_1'\underline{K} + \underline{Q}_1]\underline{z} + \dot{\underline{v}} + [\underline{A}_1'\underline{K} - \underline{K}\underline{M}]\ \underline{v} + \underline{K}\underline{e} = \underline{0} \tag{4.47}$$

If the optimal solution exists, equation (4.47) must hold for all $\underline{z}(t)$, $\underline{v}(t)$, $\underline{e}(t)$ and t. Therefore

$$\dot{\underline{K}}(t) = - \underline{K}(t)\underline{A}_1(t) - \underline{A}_1'(t)\underline{K}(t) + \underline{K}(t)\underline{M}(t)\underline{K}(t) - \underline{Q}_1 \tag{4.48}$$

and

$$\dot{\underline{v}}(t) = - [\underline{A}_1'(t)\underline{K}(t) - \underline{K}(t)\underline{M}(t)]\underline{v}(t) - \underline{K}(t)\underline{e}(t)\ . \tag{4.49}$$

The boundary conditions for (4.48) and (4.49) are found as follows. Equation (4.37) evaluated at $t=t_f$ yields

$$\underline{p}(t_f) = \underline{K}(t_f)\underline{z}(t_f) + \underline{v}(t_f)\ . \tag{4.50}$$

Equation (4.31) is repeated

$$\underline{p}(t_f) = \underline{D}_1\underline{z}(t_f). \tag{4.31}$$

Since (4.31) and (4.50) must hold for all $\underline{z}(t_f)$, it follows that

$$\underline{K}(t_f) = \underline{D}_1$$

and

$$\underline{v}(t_f) = \underline{0}. \tag{4.52}$$

Having obtained the desired control law (4.38), the partial derivatives of (4.12) can be further evaluated. Performing the multiplication indicated in (4.38) yields

$$\underline{u}^* = - \underline{R}^{-1}\underline{B}_1'[\underline{K}_1\underline{x} + \underline{K}_2\underline{s}_1 + \ldots \underline{K}_{r+1}\underline{s}_r + \underline{v}] \tag{4.53}$$

where the matrix $\underline{K}_i$ is the i^{th} (r+1)nXn column partition of $\underline{K}$. The partials of $\underline{u}$ with respect to $\underline{x}$ and $\underline{s}_j$ are

$$\partial\underline{u}/\partial\underline{x} = -\underline{R}^{-1}\underline{B}_1'\underline{K}_1 \tag{4.54}$$

and

$$\partial\underline{u}/\partial\underline{s}_j = -\underline{R}^{-1}\underline{B}_1'\underline{K}_{j+1} \tag{4.55}$$

respectively.

It should be noted that this result implies that the matrix $\underline{A}_1(t)$ in the Riccati-type differential equation (4.48) contains elements of $\underline{K}(t)$. That is from (4.15)

$$\hat{\underline{A}}(t) = \underline{A} - \underline{B}\underline{R}^{-1}\underline{B}'\underline{K}_1. \tag{4.56}$$

Thus the optimal controller utilizes state variable and sensitivity function feedback. An additional forcing function, $\underline{v}(t)$, that might be considered the sensitivity of the sensitivity function is also an input.

Figure 6 is a block diagram of the system with its feedback controller that results from the sensitivity-constrained

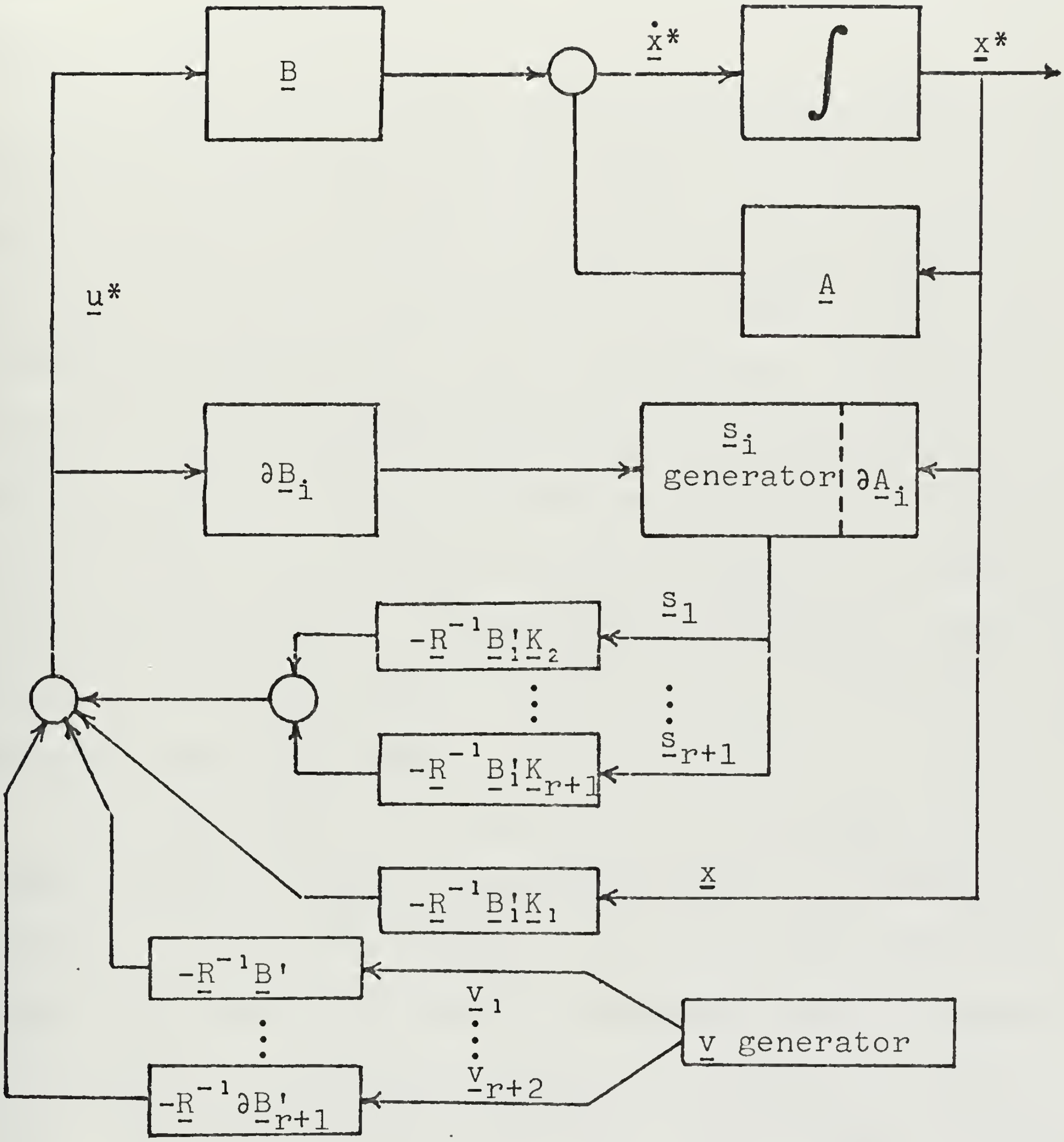

Figure 6. Block Diagram of Sensitivity Constrained Linear Regulator System.

design procedure derived above. It is clear that the derivation yields an extremely complex control strategy and any simplifications that can be made would be extremely desirable. Figure 6 assumes $\underline{v}(t)$ and $\underline{s}(t)$ are available, they must, of course, be synthesized.

The sensitivity vectors can be generated from the vector differential equation (4.13) with the state vector and control vector as forcing functions. However, the open-locp input $\underline{v}(t)$ is not so easily generated because the initial conditions for equation (4.49) are unknown. These functions depend on the partial derivatives of the sensitivity vectors with respect to q_i, the unknown parameters. The solution of (4.49) then depends on $\underline{x}(t)$ and hence is dependent on initial conditions. Consequently $\underline{v}(t)$ must be computed off line for the desired trajectory, $\underline{x}(t)$, and can be used <u>only for that trajectory</u>.

The controller described cannot be implemented in the linear regulator system in which the initial conditions are unknown. In order to implement it for a system with the trajectory, $\underline{x}(t)$, $\underline{v}(t)$ must be determined off-line by solving a two-point boundary-value problem consisting of (2r+3)n first order differential equations. Even though the equations are all linear, this is not a trivial task.

In the following sections, two design techniques that are simplifications of the above are considered. The first due to Cassidy and Lee [22] neglects the $\underline{e}_i(t)$ term of equation (4.14). The second due to Kahne [19] assumes the matrix

$\underline{B}(t)$ is not a function of $\underline{q}$. Kahne also completely neglects the entire term $\partial\underline{u}/\partial q_i$ in equation (3.84).

C. THE CASSIDY AND LEE CONTROL STRATEGY

Cassidy and Lee [22] presented a new control strategy derived from an optimization problem that considered the reduction of trajectory dispersion due to plant parameter variations. Their development was much the same as that of section B except that they neglected the effects of the term $\underline{e}(t)$ in equation (4.16) and considered only the case where there is a single input. Their results, extended to multiple inputs, are presented here as a variation of the development of section B.

The linear time-varying state regulator system is defined by equation (4.8) with its accompanying definitions

$$\dot{\underline{x}}(t) = \underline{A}(t,\underline{q})\underline{x}(t) + \underline{B}(t,\underline{q})\underline{u}(t) \qquad (4.8)$$

The sensitivity functions and equations of chapter III are repeated here for convenience,

$$\underline{s}_i = \partial\underline{x}/\partial q_i \qquad (3.83)$$

and

$$\dot{\underline{s}}_i = \partial\underline{A}_i(\underline{q})\underline{x} + \underline{A}(\underline{q})\underline{s}_i + \partial\underline{B}_i(\underline{q})\underline{u} + \underline{B}\ \partial\underline{u}/\partial q_i \qquad (3.84)$$

$$\underline{s}_i(t_o) = \underline{0}$$

where

$$\partial\underline{A}_i = \partial\underline{A}/\partial q_i \Big|_{\underline{q}=\underline{q}_o} \qquad (4.57)$$

$$\partial\underline{B}_i = \partial\underline{B}/\partial q_i \Big|_{\underline{q}=\underline{q}_o} \qquad (4.58)$$

Anticipating the resultant controller, $\underline{u}(t)$ is defined by

$$\underline{u}(t) = \underline{F}(t)\underline{x}(t) + \sum_{i=1}^{r} \underline{F}_i(t)\underline{s}_i(t). \tag{4.59}$$

Writing the partial of $\underline{u}(t)$ with respect to q_i and ignoring second partials, yields

$$\partial\underline{u}/\partial q_i = \underline{F}(t)\underline{s}_i(t), \tag{4.60}$$

where $\underline{F}(t)$ is not a function of q_i.

Substituting equation (4.60) into (3.84) yields the sensitivity equation used by Cassidy and Lee

$$\dot{\underline{s}}_i = \partial\underline{A}_i\underline{x} + (\underline{A} + \underline{B}\underline{F})\underline{s}_i + \partial\underline{B}_i\underline{u} \tag{4.61}$$

$$\underline{s}_i(t_0) = \underline{0}.$$

Again function arguments, once specified, are dropped for notational convenience.

Proceeding as in section B; adjoining the sensitivity equations to the plant equations yields the augmented system (4.15) except that the last term has been eliminated and $\hat{\underline{A}} = \underline{A} + \underline{B}\underline{F}$. Rewriting equation (4.16) with the vector $\underline{e}(t)$ eliminated yields

$$\begin{aligned} \dot{\underline{z}} &= \underline{A}_1\underline{z} + \underline{B}_1\underline{u} \\ \underline{z}(t_0) &= \{\underline{c}' \;\vdots\; \underline{0}'\}' \end{aligned} \tag{4.62}$$

where the meanings of $\underline{z}$, $\underline{A}_1$, and $\underline{B}_1$ are clear from equation (4.15) and the definition for $\hat{\underline{A}}$ above.

The performance measure for this problem is identical to the performance measure (4.18)

$$J = \tfrac{1}{2}\underline{z}'(t_f)\underline{D}_1\underline{z}(t_f) + \tfrac{1}{2}\int_{t_o}^{t_f} [\underline{z}'\underline{Q}_1\underline{z} + \underline{u}'\underline{R}\underline{u}]dt \qquad (4.18)$$

where $\underline{D}_1$, $\underline{Q}_1$, and $\underline{R}$ are as defined in section B.

The solution to the optimal control problem defined by dynamic system (4.62) and the performance measure (4.18) is well known. It can be obtained in a straightforward manner following the derivation of section B. The result is stated here.

The optimal control for (4.18) constrained by (4.62) is

$$\underline{u}^*(t) = -\underline{R}^{-1}(t)\underline{B}_1'(t)\underline{K}(t)\underline{z}(t) \qquad (4.63)$$

where $\underline{K}(t)$ is the symmetric positive definite solution to the Riccati matrix equation

$$\dot{\underline{K}} = -\underline{A}_1'\underline{K} - \underline{K}\underline{A}_1 + \underline{K}\underline{B}_1\underline{R}^{-1}\underline{B}_1'\underline{K} - \underline{Q}_1 \qquad (4.64)$$

$$\underline{K}(t_f) = \underline{D}_1.$$

The matrix $\underline{K}(t)$ is an n(r+1)X(r+1)n square matrix. If $\underline{K}(t)$ is partitioned into (r+1) matrices each of dimension n(r+1)Xn, then $\underline{K}_i$, i = 1,...,(r+1), is the i^th n(r+1)Xn column partition. Using this notation

$$\underline{u}^* = -\underline{R}^{-1}\underline{B}_1' [\underline{K}_1\underline{x} + \sum_{j=1}^{r} \underline{K}_{j+1}\, \underline{s}_j]. \qquad (4.65)$$

It should be noted that $\underline{F} = -\underline{R}^{-1}\underline{B}_1\underline{K}_1$ and consequently the matrix $\underline{A}_1$ in (4.64) is a function of elements of $\underline{K}$.

Equation (4.65) is identical to the feedback term of equation (4.53), the optimal controller for the problem of section B. The elimination of the open-loop term is an important simplification; this controller can be implemented

more easily. The block diagram of Figure 6, with the open-loop eliminated, shows the structure such a controller could take.

Kalman [3] has shown that if the pair $[A_1, B_1]$ in (4.62) is completely controllable, if $\underline{D}_1 = \underline{0}$ in (4.18), and if $\underline{A}_1, \underline{B}_1, \underline{Q}_1$, and $\underline{R}$ are constant matrices, then $\underline{K}(t) \to \underline{K}$ (a constant matrix) as $t_f \to \infty$. Clearly if these conditions are met the Cassidy and Lee result can be extended to include the infinite-time constant feedback controller. This is an important engineering result since the matrices $\underline{F}$ and $\underline{F}_j$, $j=1,\ldots,r$, in equation (4.59) are constant matrices, consequently their implementation would be very simple.

Under the conditions above, $\dot{\underline{K}} = \underline{0}$ and equation (4.64) becomes

$$\underline{0} = -\underline{A}_1'\underline{K} - \underline{K}\underline{A}_1 + \underline{K}\underline{B}_1\underline{R}^{-1}\underline{B}_1'\underline{K} - \underline{Q}_1. \qquad (4.66)$$

D. THE KAHNE CONTROL STRATEGY

In his paper "Low Sensitivity Design of Optimal Linear Control Systems" [19], Kahne has proposed a design technique for the linear regulator problem that is similar to that of the previous section. As a further simplification however, Kahne treats the problem in which the distribution matrix, $\underline{B}_1(t)$, is not a function of the parameters, $\underline{q}$. Additionally Kahne neglects the term $\partial \underline{u}/\partial q_i$ entirely. He evaluates all partial derivatives at $\underline{q}=\underline{q}_o$, the nominal parameter values. With these assumptions, the system and sensitivity constraints are defined by

$$\underline{\dot{x}}(t) = \underline{A}(t,\underline{q})\underline{x}(t) + \underline{B}(t)\underline{u}(t) \tag{4.67}$$

$$\underline{x}(t_0) = \underline{c}$$

and

$$\underline{\dot{s}}_i(t) = \partial\underline{A}_i(t,\underline{q})\underline{x}(t) + \underline{A}(t,\underline{q})\underline{s}_i(t) \tag{4.68}$$

$$\underline{s}_i(t_0) = \underline{0}$$

The sensitivity equation (4.68) is obtained by taking the partial derivative of (4.67) with respect to q_i, under the assumptions stated above. Kahne's development considered only a scalar parameter q_1. He hinted that extension to multiparameters is only a notational problem. The extension to r parameters is indicated here.

The vector $\underline{s}_i(t)$ is defined by equation (3.83); the matrix $\partial\underline{A}_i(t,\underline{q})$ is defined by

$$\partial\underline{A}_i(t,\underline{q}) = \partial\underline{A}(t,\underline{q})/\partial q_i. \tag{4.69}$$

The vectors, $\underline{x}(t)$ and $\underline{u}(t)$, and the matrices $\underline{A}(t,\underline{q})$ and $\underline{B}(t)$ are the same as those previously defined for (4.8).

The augmented system is defined as

$$\left\{\begin{matrix} \underline{\dot{x}} \\ \underline{\dot{s}}_1 \\ \cdot \\ \cdot \\ \cdot \\ \underline{\dot{s}}_r \end{matrix}\right\} = \left(\begin{matrix} \underline{A} & \underline{0} & \cdots & \underline{0} \\ \partial\underline{A}_1 & \underline{A} & \cdots & \underline{0} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \partial\underline{A}_r & \underline{0} & \cdots & \underline{A} \end{matrix}\right) \left\{\begin{matrix} \underline{x} \\ \underline{s}_1 \\ \cdot \\ \cdot \\ \cdot \\ \underline{s}_r \end{matrix}\right\} + \left\{\begin{matrix} \underline{B} \\ \underline{0} \\ \cdot \\ \cdot \\ \cdot \\ \underline{0} \end{matrix}\right\} \underline{u} \tag{4.70}$$

$$\left\{\begin{matrix} \underline{x}(t_0) \\ \underline{s}_1(t_0) \\ \vdots \\ \underline{s}_r(t_0) \end{matrix}\right\} = \left\{\begin{matrix} \underline{c} \\ \underline{0} \\ \vdots \\ \underline{0} \end{matrix}\right\}$$

which can be written more compactly as

$$\dot{\underline{z}} = \underline{A}_1\underline{z} + \underline{B}_1\underline{u}$$

$$\underline{z}(t_0) = \underline{z}_0 . \tag{4.71}$$

Kahne defines the problem for low sensitivity design as follows:

Find the admissible $\underline{u}^*(t)$ that minimizes

$$J = \tfrac{1}{2}\underline{z}'(t_f)\underline{D}_1\underline{z}(t_f) + \tfrac{1}{2}\int_{t_o}^{t_f} [\underline{z}'(t)\underline{Q}_1(t)\underline{z}(t) + \underline{u}'(t)\underline{R}(t)\underline{u}(t)]dt \tag{4.18}$$

subject to the constraints of (4.71), where $\underline{D}_1$, $\underline{Q}_1$, and $\underline{R}$ are as defined in section B.

This optimization problem has the same form as that of section A. Again the solution is well known.

The optimal controller is

$$\underline{u}^*(t) = -\underline{R}^{-1}(t)\underline{B}_1'(t)\underline{K}(t)\underline{z}(t) \tag{4.72}$$

where $\underline{K}(t)$ is the symmetric positive definite solution to the Riccati matrix differential equation

$$\dot{\underline{K}}(t) = -\underline{A}_1'(t)\underline{K}(t) - \underline{K}(t)\underline{A}_1(t) + \underline{K}(t)\underline{B}_1(t)\underline{R}^{-1}\underline{B}_1'(t)\underline{K}(t) - \underline{Q}(t) \tag{4.73}$$

$$\underline{K}(t_f) = \underline{D}_1.$$

Performing the matrix multiplication to form $\underline{B}_1'\underline{K}$ of equation (4.72) indicates that

$$\underline{B}_1'(t)\underline{K}(t) = \left\{\underline{B}'\underline{0}'\underline{0}'\ldots\underline{0}'\right\} \begin{Bmatrix} \underline{K}_{11} & \underline{K}_{12} & \cdots & \underline{K}_{1,r+1} \\ \underline{K}_{21} & \underline{K}_{22} & \cdots & \underline{K}_{2,r+1} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \underline{K}_{r+1,1} & & \cdots & \underline{K}_{r+1,r+1} \end{Bmatrix}$$

$$= \left\{ \underline{B}'\underline{K}_{11} \; \underline{B}'\underline{K}_{12} \cdots \underline{B}'\underline{K}_{1,r+1} \right\} . \qquad (4.74)$$

Using (4.74) in (4.72) yields the form of the control law

$$\underline{u}^*(t) = -\underline{R}^{-1}\underline{B}'[\underline{K}_{11}\underline{x} + \sum_{i=1}^{r} \underline{K}_{1,r+1}\, \underline{s}_i] . \qquad (4.75)$$

Note that in this formulation the matrix $\underline{A}_1$ in equation (4.73) is not a function of $\underline{K}(t)$.

As in the Cassidy and Lee formulation, if the conditions mentioned at the end of section A are met, then as $t_f \to \infty$, $\underline{K}(t) \to \underline{K}$ (a constant matrix). Then again $\dot{\underline{K}} = \underline{0}$, and equation (4.73) reduced to $[(r+1)n]^2$ nonlinear algebraic equations. However, because $\underline{K}$ is symmetric only $[(r+1)n + 1][(r+1)n]/2$ of the equations need to be solved.

The structure of the low sensitivity control system resulting from Kahne's approach is illustrated in Figure 7.

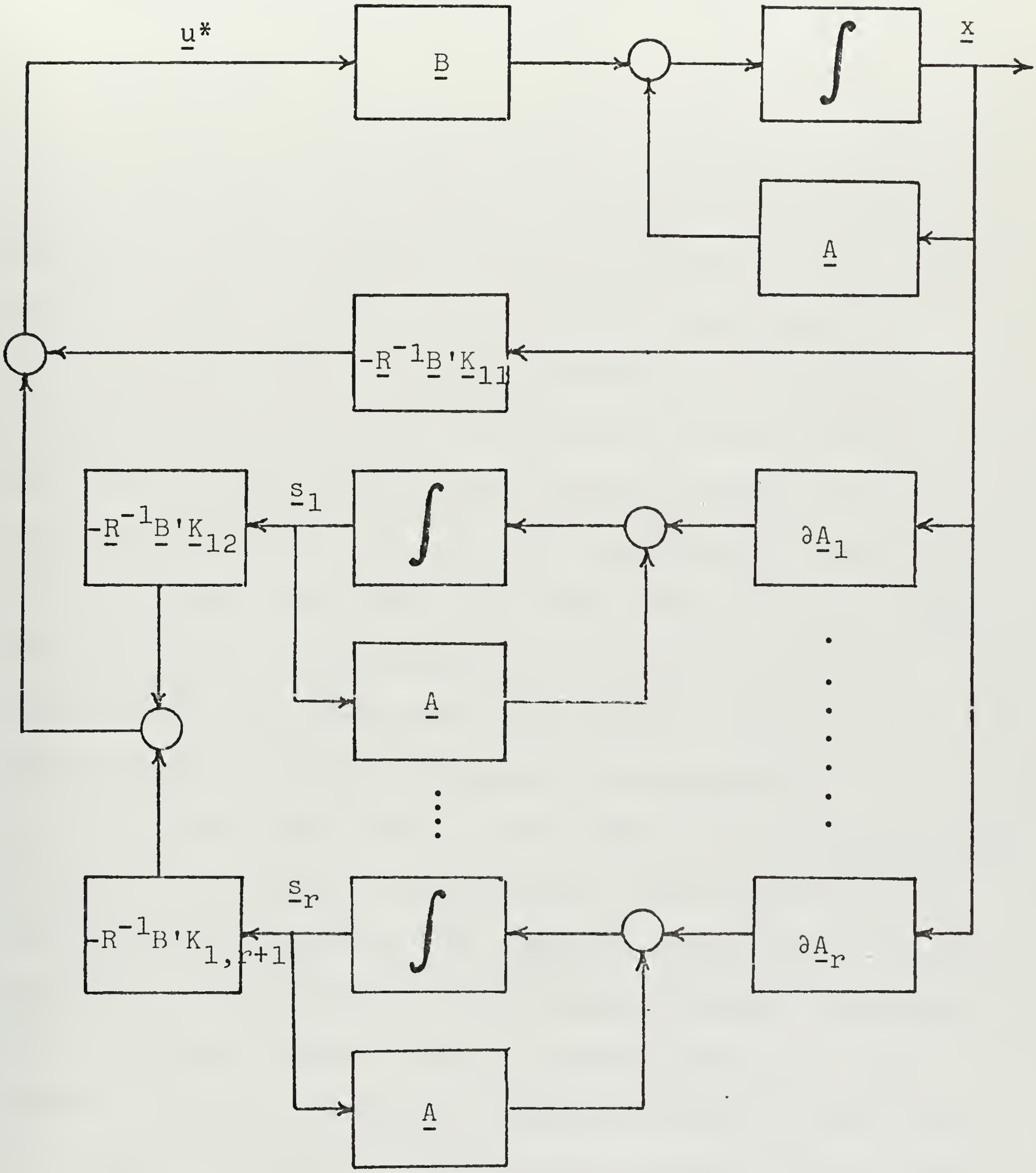

Figure 7. Block Diagram of Kahne's Low Sensitivity Linear Regulator.

V. APPLICATIONS OF SENSITIVITY DESIGN TECHNIQUES TO A FLEXIBLE SATURN BOOSTER PROBLEM

A. THE PROBLEM

In order to investigate some of the proposed low-sensitivity design strategies, a realistic problem was studied. The problem was to find a feedback control system for a large flexible booster and to apply several techniques in desensitizing its performance to parameter variations. The equations of motion and control theory applicable to the stability and response analysis for a large flexible launch vehicle were presented in detail by Garner [35] and modelled by Rillings (Saturn V-Apollo configuration) [36]. The dynamics of the Saturn V-Apollo launch vehicle was formulated and several constant-gain feedback controllers were obtained and evaluated.

It is well known that a long slender rod, unconstrained at its ends, when excited by a radial force pulse will vibrate in bending modes and at frequencies determined by its structural characteristics. The Saturn V-Apollo configuration of a launch vehicle can be characterized as a long slender rod having a length to diameter ratio of about 10:1. The flexible character of this booster system must be taken into account when designing the control system.

The large size of the Saturn V-Apollo configuration, shown in Figure 8, makes determination of the parameters describing its bending modes and frequencies extremely difficult. The parameters are generally determined by dynamic

testing, and the resulting inaccuracies must be considered. Therefore, the control system must not be sensitive to these uncertain parameters.

The problem is to design a constant-gain feedback controller that gives adequate control when the uncertain parameters differ from the nominal values by as much as 20%.

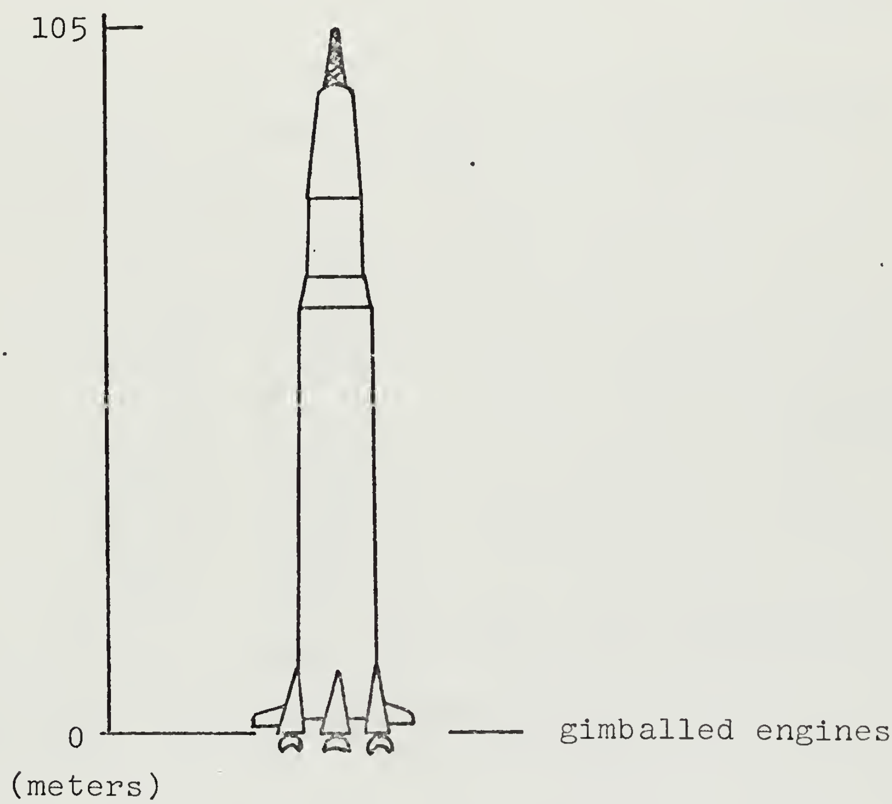

Figure 8. Vehicle Configuration.

Assuming exact knowledge of all of the states, a control that consists of gimballing the engines producing the thrust for the booster, and two frames of reference, Rillings [36] developed the following linear differential equations that describe the motion of the vehicle

$$\ddot{\phi} = -\left(\frac{R'l_{cg}}{I}\right)\beta - \left(\frac{N'l_{cp}}{I}\right)\alpha \tag{5.1}$$

$$\dot{\alpha} = -\left(\frac{T-D}{mv} - \frac{\dot{v}}{v}\right)\phi + \dot{\phi} - \left(\frac{N'}{mv} + \frac{\dot{v}}{v}\right)\alpha - \left(\frac{R'}{mv}\right)\beta \tag{5.2}$$

$$\ddot{\eta} = -2\zeta\omega\dot{\eta} - \omega^2\eta + \frac{R'Y(x_\beta)}{m}\beta \tag{5.3}$$

$$\phi_d = \phi + Y'(x_d)\eta \tag{5.4}$$

$$\dot{\phi}_r = \dot{\phi} + Y'(x_r)\dot{\eta} \tag{5.5}$$

where the variables are defined in Table 1. The state variables are defined as follows

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{Bmatrix} = \begin{Bmatrix} \phi_d \\ \dot{\phi}_r \\ \alpha \\ \eta \\ \dot{\eta} \end{Bmatrix} . \tag{5.6}$$

By substituting (5.4) and (5.5) into (5.1) and (5.2) the following state equations are obtained.

$$\dot{\phi}_d = \dot{\phi}_r + [Y'(x_d) - Y'(x_r)]\dot{\eta} \tag{5.7}$$

$$\ddot{\phi}_r = -\frac{N'l_{cp}}{I}\alpha - Y'(x_r)\omega^2\eta - 2\zeta\omega Y'(x_r)\dot{\eta}$$

$$- \left[\frac{R'l_{cg}}{I} - Y'(x_r)\frac{R'Y(x_\beta)}{m}\right]\beta \tag{5.8}$$

$$\dot{\alpha} = -\left(\frac{T-D}{mv} - \frac{\dot{v}}{v}\right)\phi_d + \dot{\phi}_r - \left(\frac{N'}{mv} + \frac{\dot{v}}{v}\right)\alpha + Y'(x_d)\left(\frac{T-D}{mv} - \frac{\dot{v}}{v}\right)\eta$$

$$- Y'(x_r)\dot{\eta} - \frac{R'}{mv}\beta \tag{5.9}$$

$$\dot{\eta} = \dot{\eta} \tag{5.10}$$

$$\ddot{\eta} = -\omega^2\eta - 2\zeta\omega\dot{\eta} + \frac{R'Y(x_\beta)}{m}\beta \tag{5.11}$$

Making the notation compatible with that of chapter IV, the control angle is defined by

$$\mu = \beta \tag{5.12}$$

The uncertain parameters to which the system is generally very sensitive are ω and $Y'(x_r)$. In the following development only the single parameter, ω, will be considered in order to limit the size of the problem.

By defining the coefficients of the matrix $\underline{A}$ and the matrix $\underline{B}$ as

$$a_1 = Y'(x_d) \qquad a_2 = Y'(x_r) \qquad a_3 = \frac{N'l_{cp}}{I}$$

$$a_4 = 2\zeta \qquad a_5 = \frac{T-D}{mv} - \frac{\dot{v}}{v} \qquad a_6 = \frac{N'}{mv} + \frac{\dot{v}}{v}$$

$$b_1 = \frac{R'l_{cg}}{I} - a_2 b_3 \qquad b_2 = \frac{R'}{mv} \qquad b_3 = \frac{R'Y(x_\beta)}{m}$$

equations (5.7) - (5.11) can be written in state variable form as

$$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x} + \underline{b}\mu \tag{5.13}$$

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{Bmatrix} = \begin{Bmatrix} 0 & 1 & 0 & 0 & a_1 - a_2 \\ 0 & 0 & -a_3 & -a_2\omega^2 & -a_4 a_2 \omega \\ -a_5 & 1 & -a_6 & a_1 a_5 & -a_2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\omega^2 & -a_4\omega \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -b_1 \\ -b_2 \\ 0 \\ b_3 \end{Bmatrix} \mu \tag{5.14}$$

TABLE I

Nomenclature

ϕ	pitch angle
α	angle of attack
η	bending mode deflection
T	thrust
D	drag
m	mass
I	moment of inertia about C.G. (center of gravity
l_{cg}	distance from gimbal to C.G.
l_{cp}	distance from C.G. to C.P. (center of pressure
v	vehicle velocity
R'	thrust of gimballed engine
N'	normal aerodynamic force coefficient
$Y(x_\beta)$	deflection mode shape at the gimbal
ζ	bending mode damping
ω	bending mode frequency
$Y'(x_d)\eta$	displacement at the pitch angle gyro due to the bending mode
$Y'(x_r)\dot{\eta}$	angular rate at the pitch angle rate gyro due to the bending mode
ϕ_d	pitch angle gyro output
$\dot{\phi}_r$	pitch angle rate gyro output
β	gimball engine angular deflection

B. OPTIMAL SOLUTION

All of the parameters in $\underline{A}$ and $\underline{b}$ vary considerably during the launch phase as functions of time. However, in order to simplify the example a frozen-time-point model of the vehicle was used. The control system for the time-invariant system should be designed for the most critical time of the launch phase. For this problem the time t = 80 seconds was chosen. At this time the vehicle is subjected to extreme aerodynamic forces and wind disturbances. Zero problem time in subsequent solutions corresponds to flight time, t = 80 seconds. At t = 80 seconds the coefficients a_i have the numerical values [36]:

$$a_1 = 1.5X10^{-2} \qquad a_2 = 7.0X10^{-3} \qquad a_3 = -2.03X10^{-1}$$

$$a_4 = 2.00X10^{-2} \qquad a_5 = 1.37X10^{-2} \qquad a_6 = 4.07X10^{-2}$$

$$b_1 = -6.15X10^{-1} \qquad b_2 = 3.34X10^{-2} \qquad b_3 = 2.55X10^{2}.$$

The $\underline{A}$ matrix of equation (5.13) becomes

$$\underline{A} = \begin{Bmatrix} 0 & 1. & 0 & 0 & 8.0X10^{-3} \\ 0 & 0 & 2.03X10^{-1} & -3.13X10^{-1} & -9.36X10^{-4} \\ -1.37X10^{-2} & 1. & -4.07X10^{-2} & 2.05X10^{-4} & -7.0X10^{-3} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -4.47X10^{1} & -1.34X10^{-1} \end{Bmatrix} \tag{5.15}$$

where $\omega_o = 6.68$. The $\underline{b}$ vector becomes

$$\underline{b} = \begin{Bmatrix} 0 \\ 6.15X10^{-1} \\ -3.34X10^{-2} \\ 0 \\ 2.55X10^{+2} \end{Bmatrix} . \tag{5.16}$$

The pair $[\underline{A},\underline{b}]$ above are completely controllable; therefore, the optimal control

$$\mu^*(t) = -\frac{1}{\rho}\underline{b}'\underline{K}\underline{x}(t) \tag{5.17}$$

or

$$\mu^*(t) = \underline{f}'\underline{x}(t) \tag{5.18}$$

that minimizes the performance measure

$$J = \tfrac{1}{2}\int_0^\infty [\underline{x}'\underline{Q}\underline{x} + \rho\mu^2]dt \tag{5.19}$$

has a constant gain vector $\underline{f}'$.

The optimal control for this system was obtained by integrating the Riccati equation

$$\dot{\underline{K}} = -\underline{A}'\underline{K} - \underline{K}\underline{A} + \underline{K}\underline{B}\,\frac{1}{\rho}\,\underline{B}'\underline{K} - \underline{Q} \tag{5.20}$$

backwards in time until a steady state solution was obtained using a 4^{th}-order Runge-Kutta[1] integration scheme. Although $\underline{K}$ is a 5X5 matrix, only 15 equations were solved due to symmetry. The constant-gain vector is given by

$$\underline{f}' = -\frac{1}{\rho}\,\underline{b}'\underline{K}\ . \tag{5.21}$$

The matrix $\underline{Q}$ and weighting factor used in the optimal solution were

$$\underline{Q} = \begin{Bmatrix} .75 & 0 & 0 & 0 & 0 \\ 0 & .01 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{Bmatrix} \tag{5.22}$$

---

[1] Subroutine RKLDEQ, NPS Computer Facility.

and

$$\rho = 1. \tag{5.23}$$

The optimal control with $\omega = \omega_0$ was found to be

$$\underline{f}' = \{1.23 \quad 1.98 \quad .976 \quad -.0229 \quad -.0153\}. \tag{5.24}$$

The optimal trajectories, with $\omega = \omega_0$, for x_1, x_2, x_3, and x_4 versus time are shown in Figure 9. These curves were obtained using an initial condition pitch angle rate of 5°/sec. to simulate a severe wind gust. The pitch angle, x_1 is seen to damp out quite rapidly, however there is considerable oscillation at the natural bending mode frequency. The bending mode deflection, x_4, measured at the gimbal station, is large but damps out quite rapidly. The angle of attack stays small and returns close to zero in about three seconds. The angle of attack then remains at a small acceptable negative value for quite a long time.

In order to observe the effects of parameter variations on the system, the normalized trajectory error was defined as follows. Let x_i be the trajectory determined using the perturbed parameter, ω, and let x_{io} be the trajectory determined using the nominal parameter, ω_0, then the normalized error, e_i, is defined by

$$e_i = \frac{x_i - x_{io}}{\omega - \omega_0}\,. \tag{5.25}$$

In determining sensitivity in this way, there is no question about sensitivity model approximations, or concern about excluded terms.

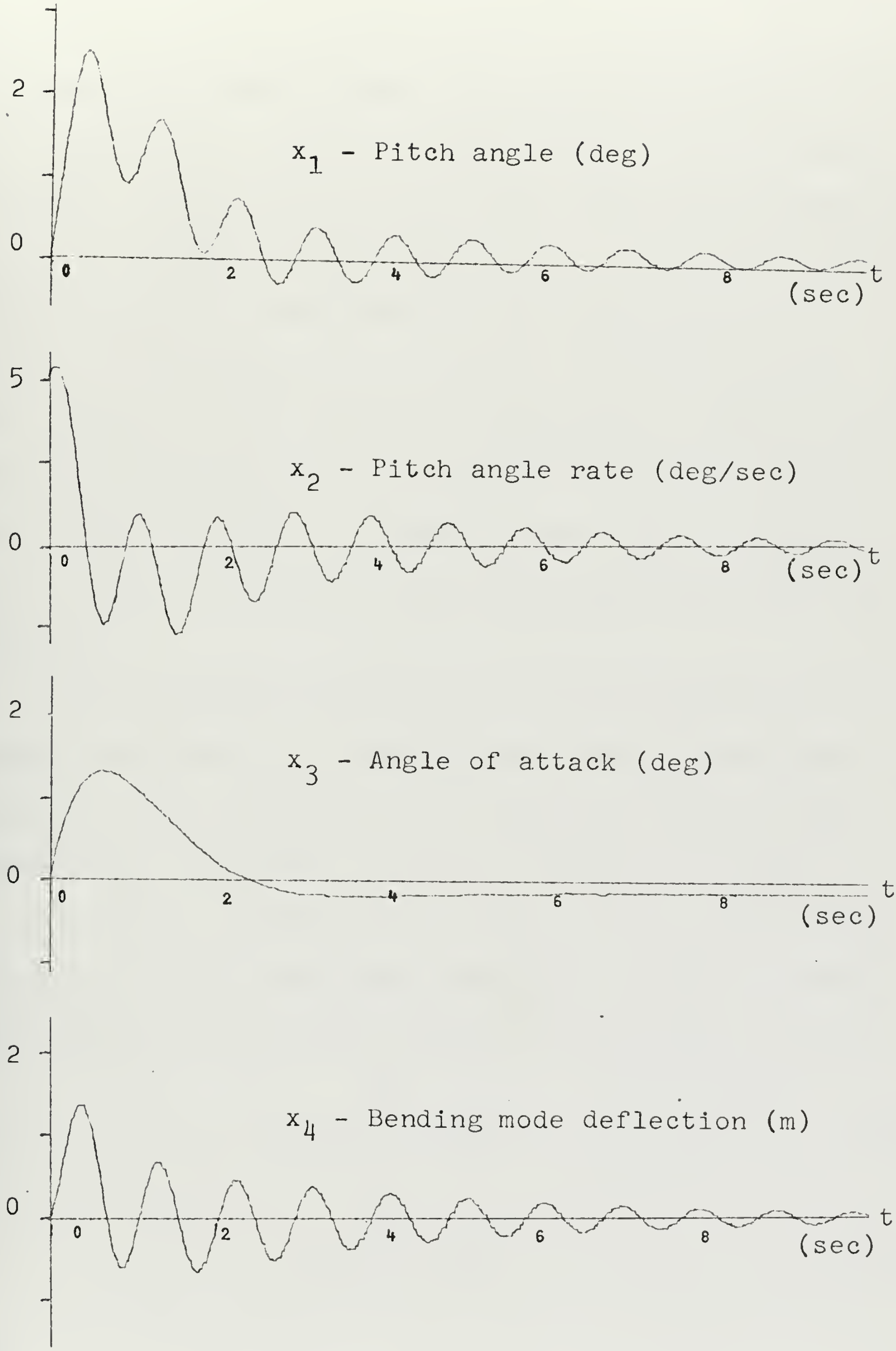

Figure 9. Optimal Control; $\omega = \omega_o$; $Y = Y_o$.

Error trajectories for pitch angle, e_1, and bending mode deflections, e_4, along with the control, μ, are shown in Figure 10. These curves were generated by using an initial condition of 5°/sec. for pitch rate. The perturbed parameter was $\omega = 0.8\ \omega_o$. The pitch angle error trajectory e_1, is oscillatory at the frequency, ω. It reaches a peak value of about 1.2 and decays slowly to about 0.4 after 6 seconds. The bending mode deflection error trajectory, e_4, was observed to have about the same general characteristics. The control history had a large initial value of about 10° which decayed rapidly to a small value. The coupling of the bending mode frequency onto the control was small but noticable.

These normalized error results were difficult to analyze meaningfully. They indicated that if parameter variations were increased, that the error would be increased also, but this was already known from (5.25).

In order to establish a measure for the sensitivity of a system which provided a meaningful basis for comparison, the following integral squared errors were defined,

$$J_{\underline{x}} = \int_0^{t_f} \underline{x}'\underline{x}\,dt \tag{5.26}$$

$$J_{\underline{s}} = \int_0^{t_f} \underline{s}'\underline{s}\,dt \tag{5.27}$$

and

$$J_{\mu} = \int_0^{t_f} \rho\mu^2\,dt. \tag{5.28}$$

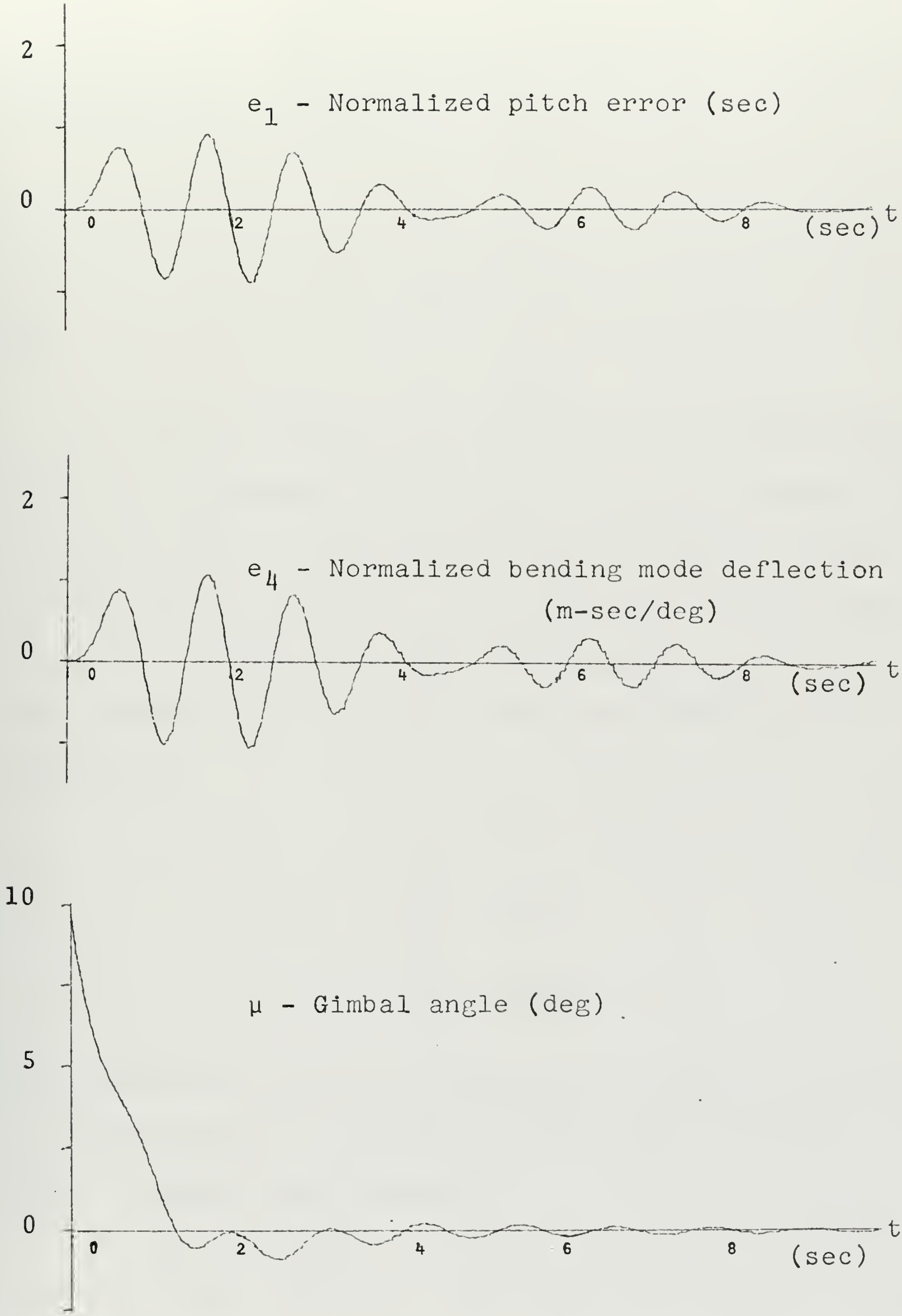

Figure 10. Optimal Control: $\Delta\omega=\omega_o - 0.8\ \omega_o$; $Y = Y_o$.

The vector, $\underline{s}$, in (5.27) was the same as that defined by equation (3.83). For the optimal controller, and nominal plant, with t_f = 20 seconds, the values were

$$J_{\underline{x}} = 41.0$$

$$J_{\underline{s}} = 1{,}550$$

and

$$J_{\mu} = .0393.$$

In order to obtain more information about the sensitivity of the system to variations in ω, the response of the optimal system was obtained for $\omega = 0.8\omega_o$. The resulting trajectories for x_1, x_2, x_3, and x_4 versus time are shown in Figure 11. Additionally, $J_{\underline{x}}$, $J_{\underline{s}}$, and J_{μ} were computed for this case. The result with t_f = 20 seconds was

$$J_{\underline{x}} = 66.1$$

$$J_{\underline{s}} = 146.4$$

$$J_{\mu} = 0.0392.$$

The system was excessively sensitive to a 20% variation in ω. Additionally, the maximum bending mode deflection of about 2.0 meters was excessive.

In order to reduce the sensitivity several proposed techniques were utilized. The results are reported in the following sections.

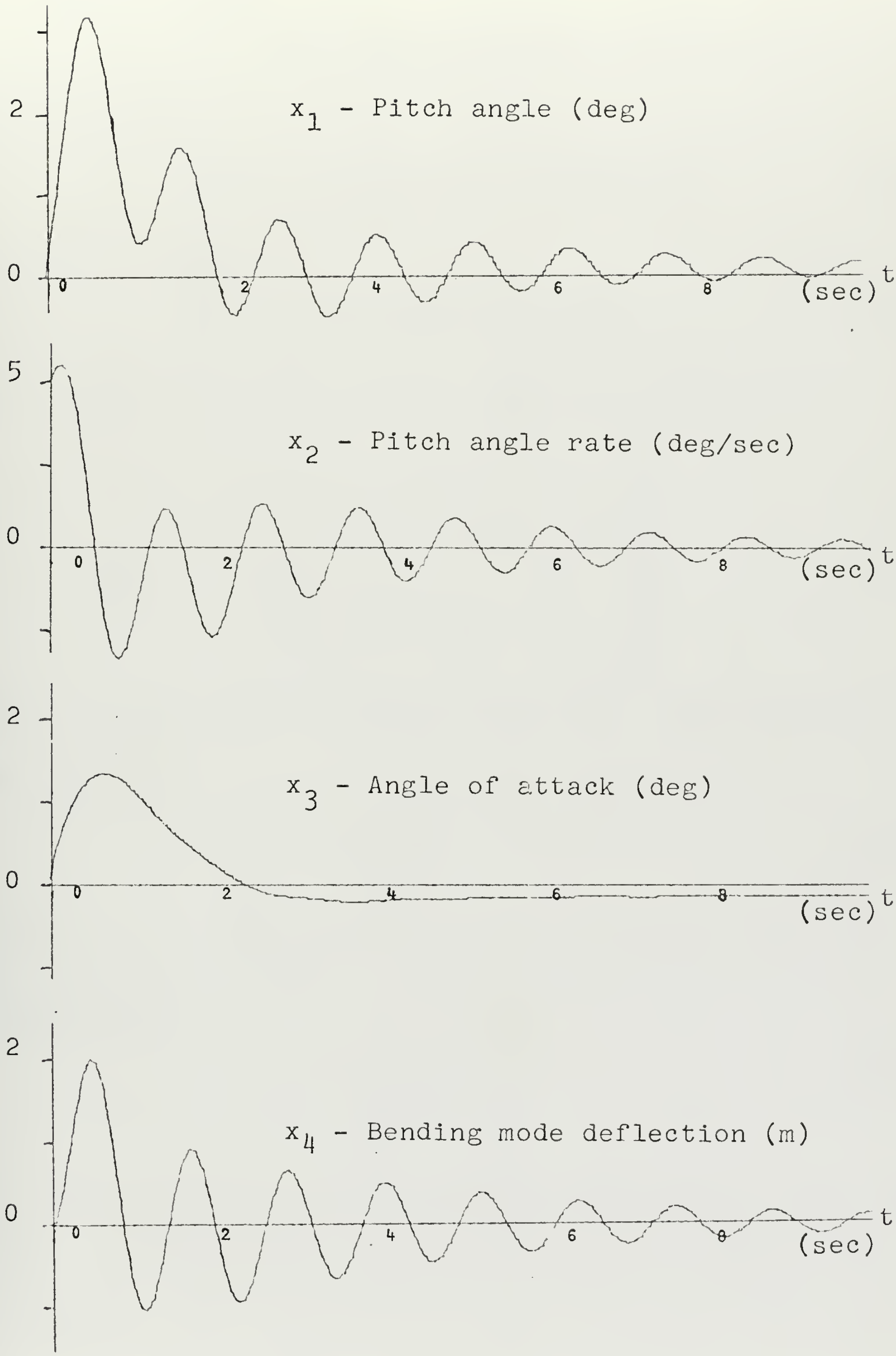

Figure 11. Optimal Control: $\omega = 0.8\ \omega_o$; $Y = Y_o$.

C. KAHNE'S METHOD

By augmenting the system equation (5.13) with the sensitivity equation

$$\dot{\underline{s}}_1(t) = \partial\underline{A}_1(\omega)\underline{x}(t) + \underline{A}(\omega)\underline{s}_1(t) \qquad (5.29)$$

the augmented system

$$\dot{\underline{z}}(t) = \underline{A}_1(\omega)\underline{z}(t) + \underline{b}_1\mu(t) \qquad (5.30)$$

was obtained. The matrix $\partial\underline{A}_1 = \partial\underline{A}/\partial\omega$, was defined by

$$\partial\underline{A}_1 = \begin{Bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2a_2\omega & -a_2a_4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\omega & -a_4 \end{Bmatrix} \qquad (5.31)$$

where the matrix $\underline{A}$ was defined by (5.14). The vector $\underline{b}_1$ was defined by

$$\underline{b}_1 = \begin{Bmatrix} \underline{b} \\ \hline \underline{0} \end{Bmatrix} \qquad (5.32)$$

The matrix $\underline{A}_1$ of equation (5.30) was defined by the partitioned matrix

$$\underline{A}_1 = \left\{ \begin{array}{c:c} \underline{A} & \underline{0} \\ \hdashline \partial\underline{A}_1 & \underline{A} \end{array} \right\} \qquad (5.33)$$

The numerical values for $\underline{A}$ and $\underline{b}$ were defined by (5.15) and (5.16). The numerical values for the matrix $\partial\underline{A}_1$ were the following

$$\partial \underline{A}_1 = \begin{Bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -9.36\text{X}10^{-2} & -1.4\text{X}10^{-4} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.34\text{X}10^{+1} & -2.0\text{X}10^{-2} \end{Bmatrix} . \quad (5.34)$$

The pair $[\underline{A}_1, \underline{b}_1]$ were not completely controllable for this case. Under these circumstances, the problem could not be cast as an infinite-interval process, hence there was no guarantee that a solution with constant-gain feedback existed. Following a suggestion by Kirk [2], the problem was cast as a finite time problem of long duration. The problem was defined to find $\mu(t) = \underline{f}'(t)\underline{z}(t)$ such that the performance measure

$$J + \tfrac{1}{2}\underline{z}'(t_f)\underline{D}_1\underline{z}(t_f) + \tfrac{1}{2}\int_0^{t_f} [\underline{x}'\underline{Q}\underline{x} + \underline{s}'\underline{W}\underline{s} + \rho\mu^2]dt \quad (5.35)$$

was minimized. The final time t_f was chosen to be 200 seconds and the value of the weighting matrix was $\underline{D}_1 = \underline{0}$.

The weighting matrix $\underline{Q}$ and scalar ρ were defined by (5.22) and (5.23) respectively. The matrix $\underline{W}$ was defined by

$$\underline{W} = \begin{Bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .01 & 0 \\ 0 & 0 & 0 & 0 & .01 \end{Bmatrix} . \quad (5.36)$$

The Riccati equations were integrated backwards in time and a nearly steady state solution was reached. Since t_f =

200 seconds was much longer than the interval of interest, the time-varying gains were approximated by

$$\underline{f}' = -\frac{1}{\rho}\underline{b}_1'\underline{K}(0) \ . \tag{5.37}$$

It should be noted that obtaining the above solution required the solution of 55 nonlinear differential equations, where the advantage of the symmetry of $\underline{K}(t)$ was used.

The numerical values thus obtained for the feedback gains were

$$\begin{aligned} f' = \{&1.25 \quad 2.54 \quad 1.06 \quad -.287 \quad -.0526 \\ &\qquad\quad 0 \qquad 0 \qquad 0 \quad -.494 \quad -.0636\}. \end{aligned} \tag{5.38}$$

The trajectories for pitch angle, x_1, pitch angle rate, x_2, angle of attack, x_3, and bending mode deflection, x_4, for $\omega = \omega_o$, are shown on Figure 12. The oscillations which were present in Figure 9 are gone. The peak value of pitch angle has increased to about 3°. The over-shoot of angle of attack is also increased. The maximum bending mode deflection however has been reduced to about half of the value obtained with the optimal controller, and the oscillations at the bending mode frequency have almost been eliminated.

Figure 13 shows the sensitivity functions, $s_1 = \partial x_1/\partial\omega$, $s_4 = \partial x_4/\partial\omega$ and the control history, μ, for $\omega = \omega_o$. These sensitivity terms were reduced considerably compared to the optimal solution with no sensitivity considerations included.

The values of $J_{\underline{x}}$, $J_{\underline{s}}$, and J_μ, with $t_f = 20$ sec. and $\omega = \omega_o$ were

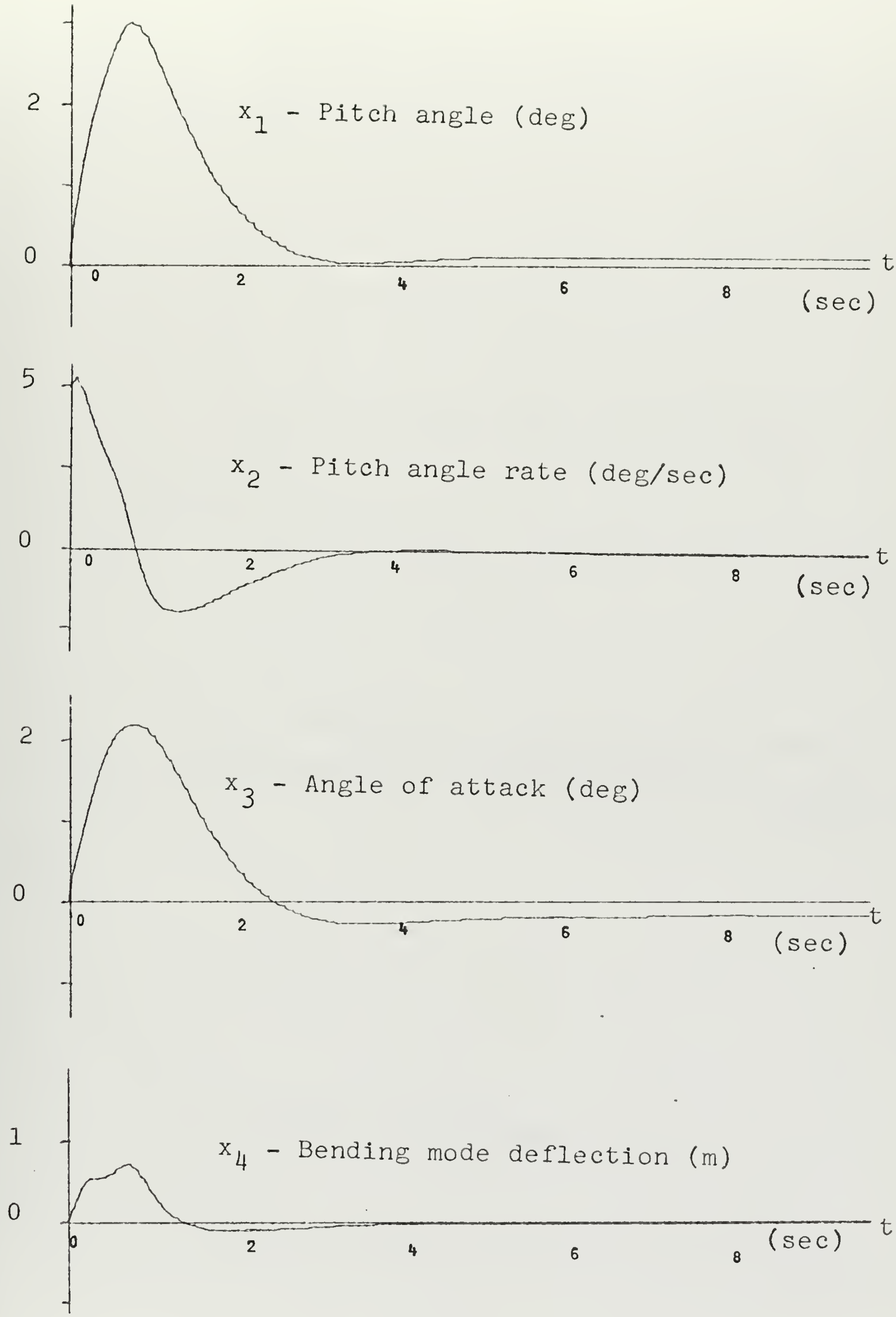

Figure 12. Kahne Control: $\omega = \omega_o$; $Y = Y_o$.

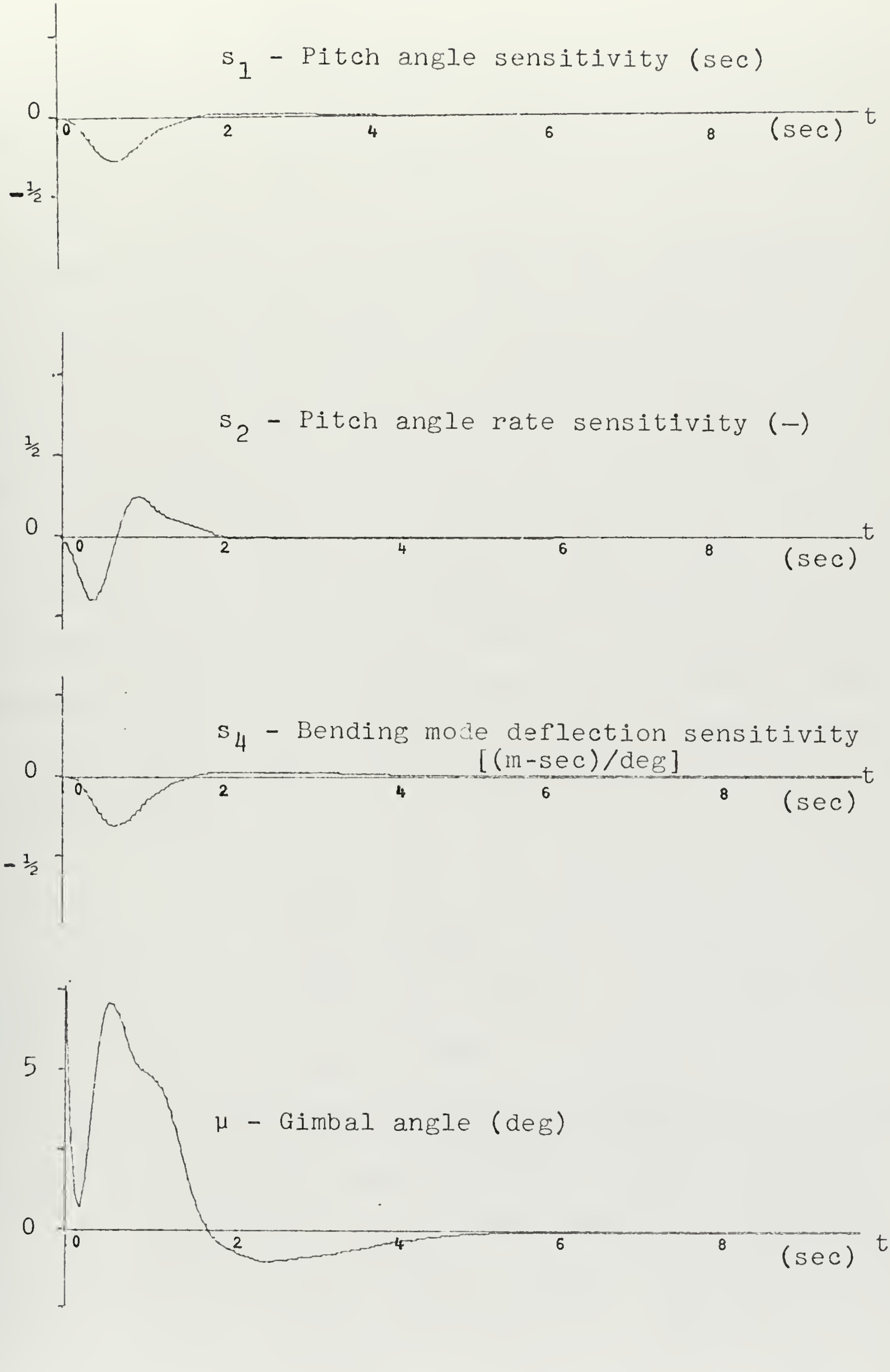

Figure 13. Kahne Control: $\omega = \omega_o$; $Y = Y_o$.

$$J_{\underline{x}} = 2.64$$

$$J_{\underline{s}} = .423$$

and

$$J_{\mu} = .0596.$$

Figure 14 shows the trajectories with the parameter, $\omega = 0.8\omega_o$. The trajectories x_1 and x_3 changed less than 10% from nominal at their maximum value, however, x_4 increases by almost 65%. This was a greater percentage change than the approximately 45% change that occurred with the same parameter variation for the optimal system. However, in this case even with a parameter change of 20%, the maximum bending mode deflection was about 1.2 meters which is considerably less than the 2 meters for the optimal case.

The values of $J_{\underline{x}}$, $J_{\underline{s}}$, and J_{μ}, with $t_f = 20$ sec. and $= 0.8\omega_o$ were

$$J_{\underline{x}} = 3.82$$

$$J_{\underline{s}} = .538$$

and

$$J_{\mu} = .0591 .$$

Figure 15 shows the sensitivity functions, $s_1 = \partial x_1/\partial\omega$, $s_4 = \partial x_4/\partial\omega$, and the control history, μ, for $\omega = 0.8\omega_o$.

The system with Kahne's controller was considerably less sensitive than the optimal system. In order to improve the sensitivity a trade-off between the variations of x_1, x_2, and x_3 and the variations of x_4 and x_5 and the sensitivity trajectories occurred. There was large variation in the control

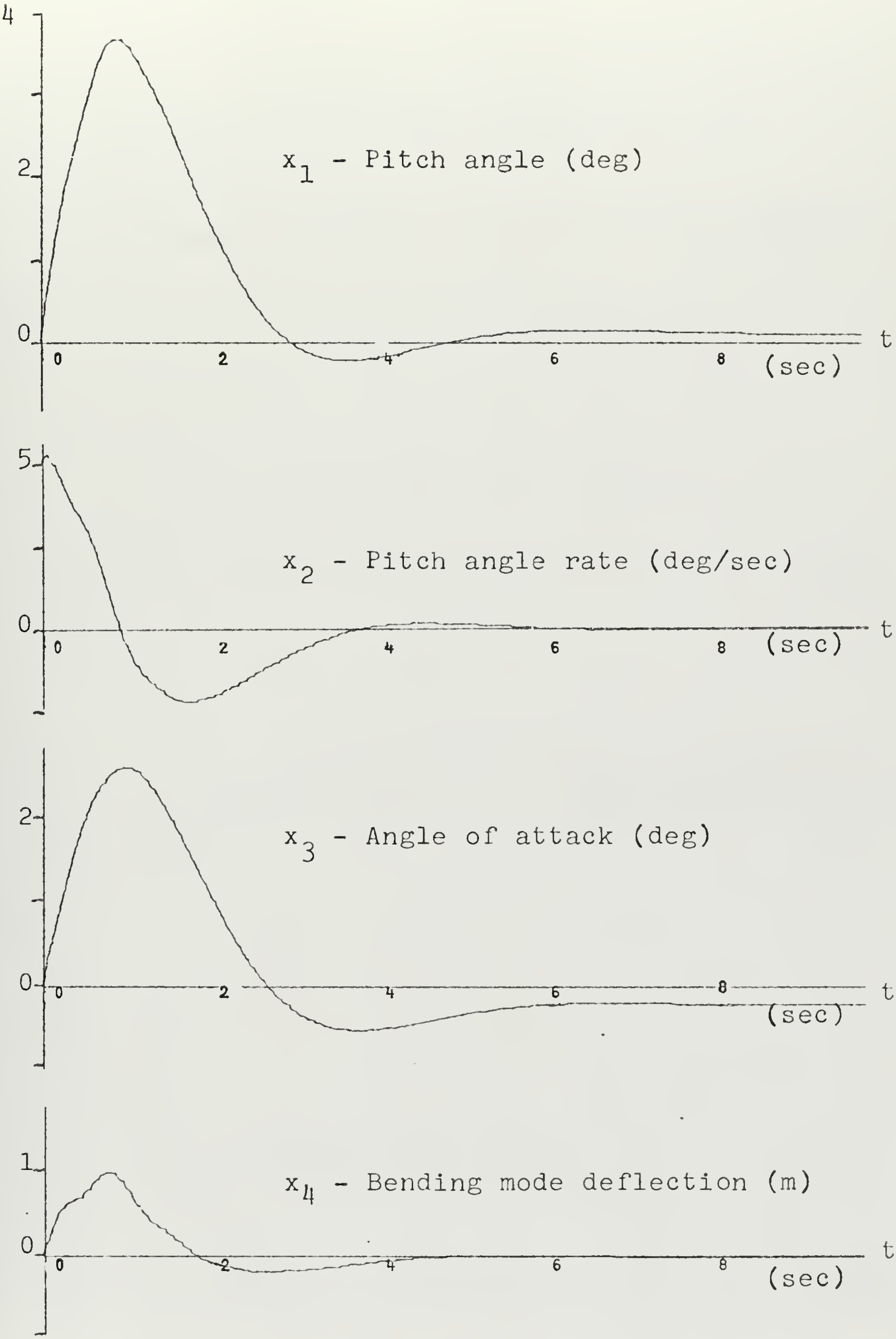

Figure 14. Kahne Control: $\omega=0.8\omega_o$; $Y = Y_o$.

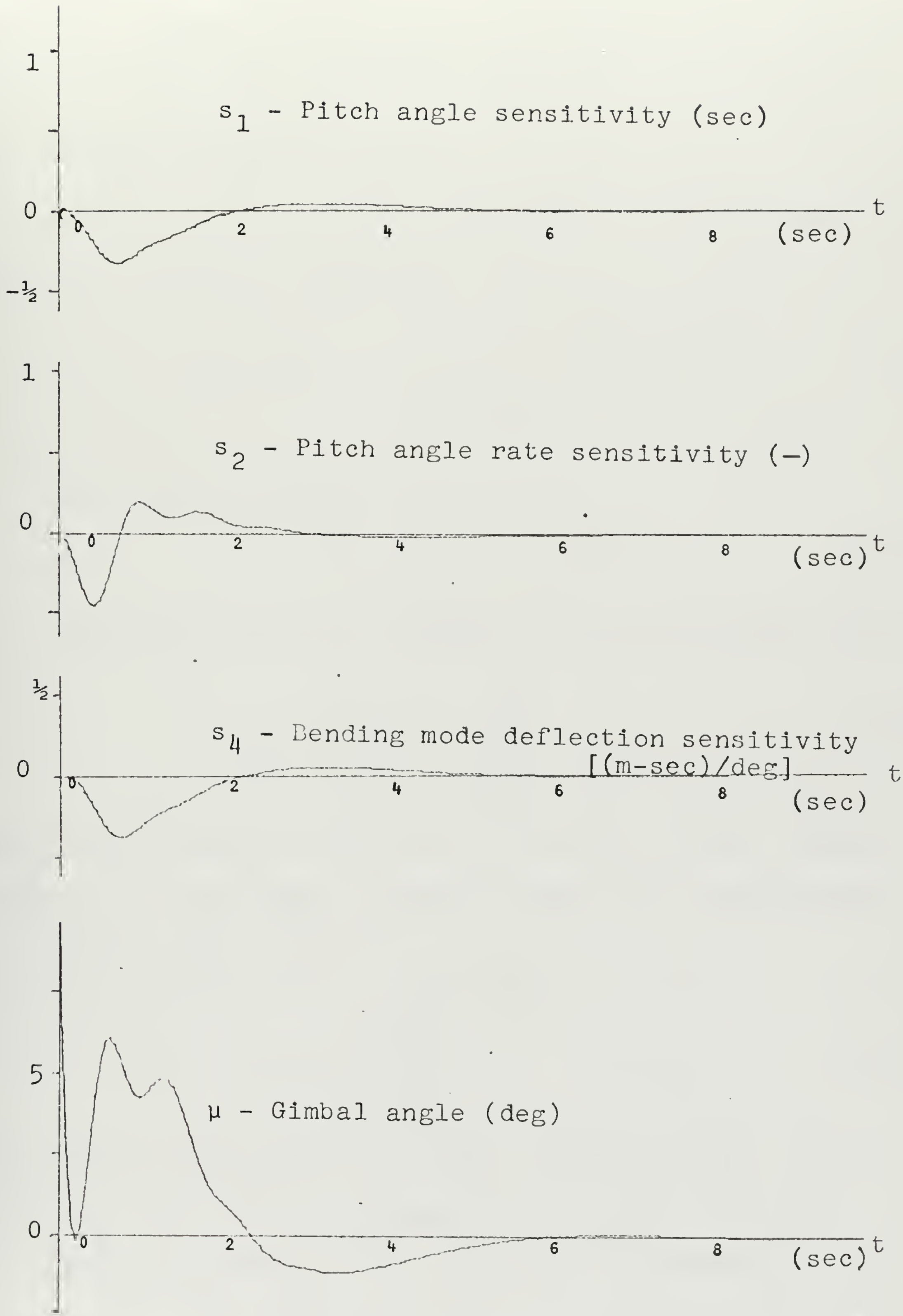

Figure 15. Kahne Control: $\omega = 0.8\omega_o$; $Y = Y_o$.

magnitudes. Kahne's controller also resulted in a considerable change in the oscillation frequency and magnitude of the bending mode deflection which was a desired result.

D. CASSIDY AND LEE'S METHOD

In this case the system equation (5.13) was augmented with the sensitivity equation

$$\dot{\underline{s}}_1(t) = \partial\underline{A}_1(\omega)\underline{x}(t) + \underline{A}(\omega)\underline{s}_1(t) + \partial\underline{B}_1(\omega)\mu(t) \qquad (5.39)$$

$$\underline{s}_1(0) = \underline{0}.$$

The augmented system equation was

$$\dot{\underline{z}}(t) = \underline{A}_1(\omega)\underline{z}(t) + \underline{b}_1\mu(t). \qquad (5.40)$$

The matrix $\underline{A}_1(\omega)$ was defined by the partitioned matrix

$$\underline{A}_1 = \left\{\begin{array}{c:c} \underline{A} & \underline{0} \\ \hdashline \partial\underline{A}_1 & \hat{\underline{A}} \end{array}\right\} \qquad (5.41)$$

where $\underline{A}$ and $\partial\underline{A}_1$ were defined by (5.14) and (5.31), respectively. In this case the vector $\underline{b}$ was not a function of ω, therefore $\partial\underline{B}_1 = \underline{0}$ and

$$b_1 = \left\{\begin{array}{c} \underline{b} \\ \cdots \\ \underline{0} \end{array}\right\}. \qquad (5.42)$$

The matrix $\underline{A}$ from equation (4.61) was $\hat{\underline{A}} = \underline{A} + \underline{b}\underline{f}'$. Expressing $\hat{\underline{A}}$ in terms of elements of $\underline{K}$ instead of $\underline{f}$, $\hat{\underline{A}} = \underline{A} - 1/\rho\ \ \underline{b}\underline{b}'\underline{K}_{11}$ which is the applicable expression obtained from (4.65). The nXn matrix $\underline{K}_{11}$ was defined by the partitioned matrix

$$\underline{K} = \left\{ \begin{array}{c:c} \underline{K}_{11} & \underline{K}_{12} \\ \hdashline \underline{K}_{21} & \underline{K}_{22} \end{array} \right\} . \tag{5.43}$$

The numerical values for $\underline{A}$, $\partial\underline{A}_1$, and $\underline{b}$ were defined by (5.15), (5.16), and (5.34) respectively. The numerical values for $\hat{\underline{A}}$ remain to be determined.

The problem was to find the $\mu^* = \underline{f}'\underline{z}$ that minimized the performance measure (5.35) with weighting matrices $\underline{Q}$ and $\underline{W}$ and the weighting scalar ρ defined by (5.22), (5.36), and (5.23).

The modified 2nX2n matrix Riccati equation

$$\dot{\underline{K}} = -\left\{ \begin{array}{c:c} \underline{A}' & \partial\underline{A}_1' \\ \hdashline \underline{0} & \underline{A}' - \underline{K}_{11}\underline{b}\underline{b}' \end{array} \right\} \left\{ \begin{array}{c:c} \underline{K}_{11} & \underline{K}_{12} \\ \hdashline \underline{K}_{21} & \underline{K}_{22} \end{array} \right\} - \left\{ \begin{array}{c:c} \underline{K}_{11} & \underline{K}_{12} \\ \hdashline \underline{K}_{21} & \underline{K}_{22} \end{array} \right\} \cdot \left\{ \begin{array}{c:c} \underline{A} & \underline{0} \\ \hdashline \partial\underline{A}_1 & \underline{A} - \underline{b}\underline{b}'\underline{K}_{11} \end{array} \right\} + \underline{K}\underline{b}\underline{b}'\underline{K} - \left\{ \begin{array}{c:c} \underline{Q} & \underline{0} \\ \hdashline \underline{0} & \underline{W} \end{array} \right\} \tag{5.44}$$

$$\underline{K}(t_f) = \underline{0}$$

was integrated backward in time until a steady-state solution was obtained.

The numerical values obtained from $\underline{f}' = -\frac{1}{\rho}\underline{b}'\underline{K}_{11}$ were

$$\begin{aligned} \underline{f}' = \{ & 1.23 \quad 2.28 \quad 1.03 \quad -.124 \quad -.0352 \\ & 1.33 \quad 2.28 \quad 1.07 \quad -.272 \quad -.0092 \} . \end{aligned} \tag{5.45}$$

In this case with the numerical values of $\hat{\underline{A}}$ determined, the pair $[\underline{A}, \underline{b}]$ were completely controllable, therefore the conditions for the infinite-interval regulator problem to have a steady state solution were satisfied.

Figures 16 and 17 show the Cassidy and Lee control system trajectories x_1, x_2, x_3, x_4, s_1, s_2, s_4, and μ versus time for $\omega = \omega_o$. Comparing them to Figures 12 and 13 (Kahne control system) showed small differences. Figures 18 and 19 for the Cassidy and Lee control system, with $\omega = 0.8\omega_o$, when compared with Figures 14 and 15, the comparable trajectories for the Kahne control system, also showed small differences. One concludes that the two methods were approximately equivalent in sensitivity reduction and control. Comparing $J_{\underline{x}}$, $J_{\underline{s}}$, and J_μ between the two systems indicated that the Cassidy and Lee controller led to a less sensitive system but at a greater cost in $J_{\underline{x}}$. The values $J_{\underline{x}}$, $J_{\underline{s}}$, and J_μ, with t_f = 20 sec. and $\omega = \omega_o$ were

$$J_{\underline{x}} = 3.786$$

$$J_{\underline{s}} = .2909$$

and

$$J_\mu = .0491.$$

For $\omega = 0.8\omega_o$ the values were

$$J_{\underline{x}} = 5.727$$

$$J_{\underline{s}} = .4758$$

and

$$J_\mu = .0488.$$

Of course, all of the results obtained above were directly related to the weighting matrices $\underline{Q}$ and $\underline{W}$ used in the problem solutions. Both of these matrices were obtained by trial-and-error techniques.

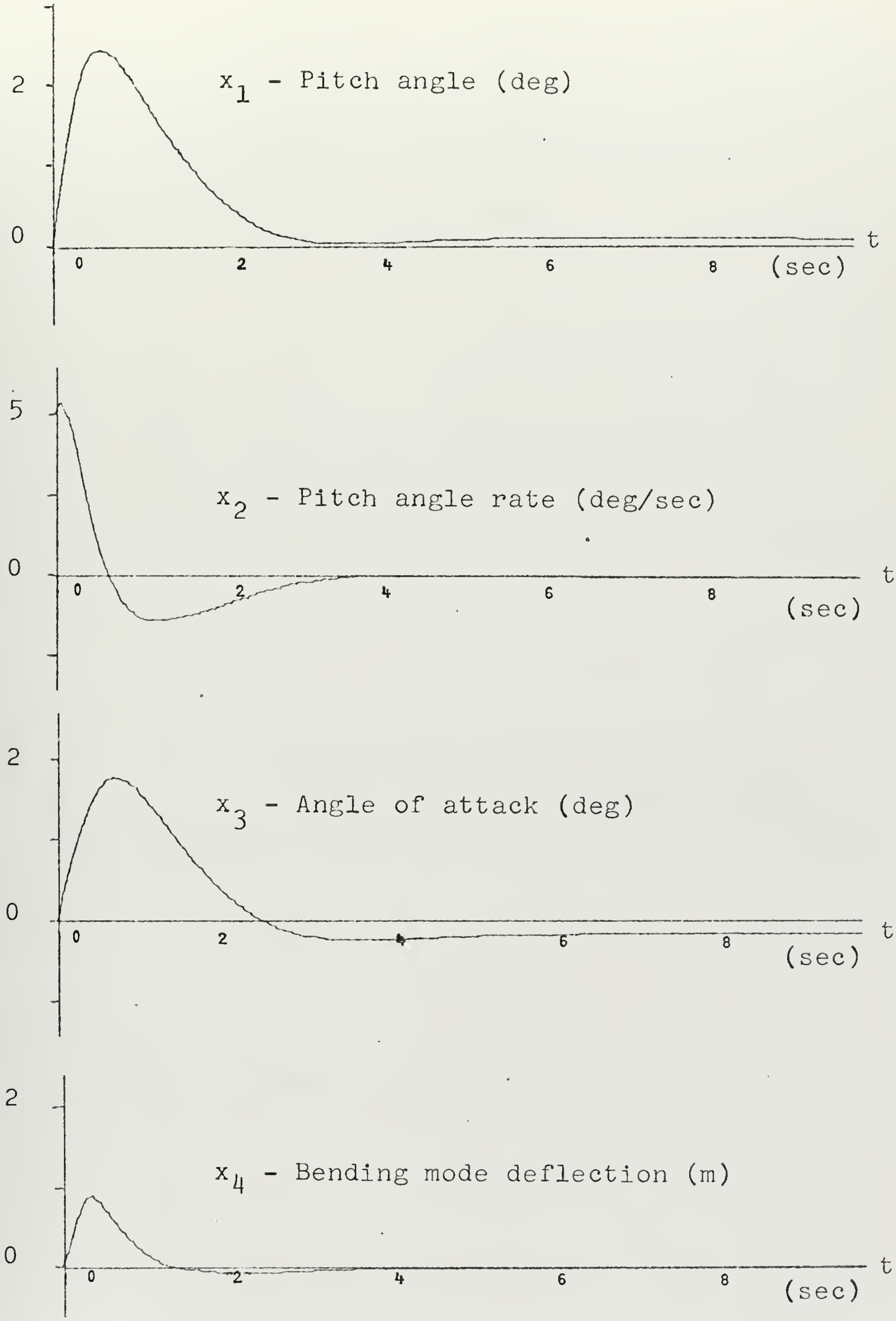

Figure 16. Cassidy and Lee Control: $\omega=\omega_o$; $Y = Y_o$.

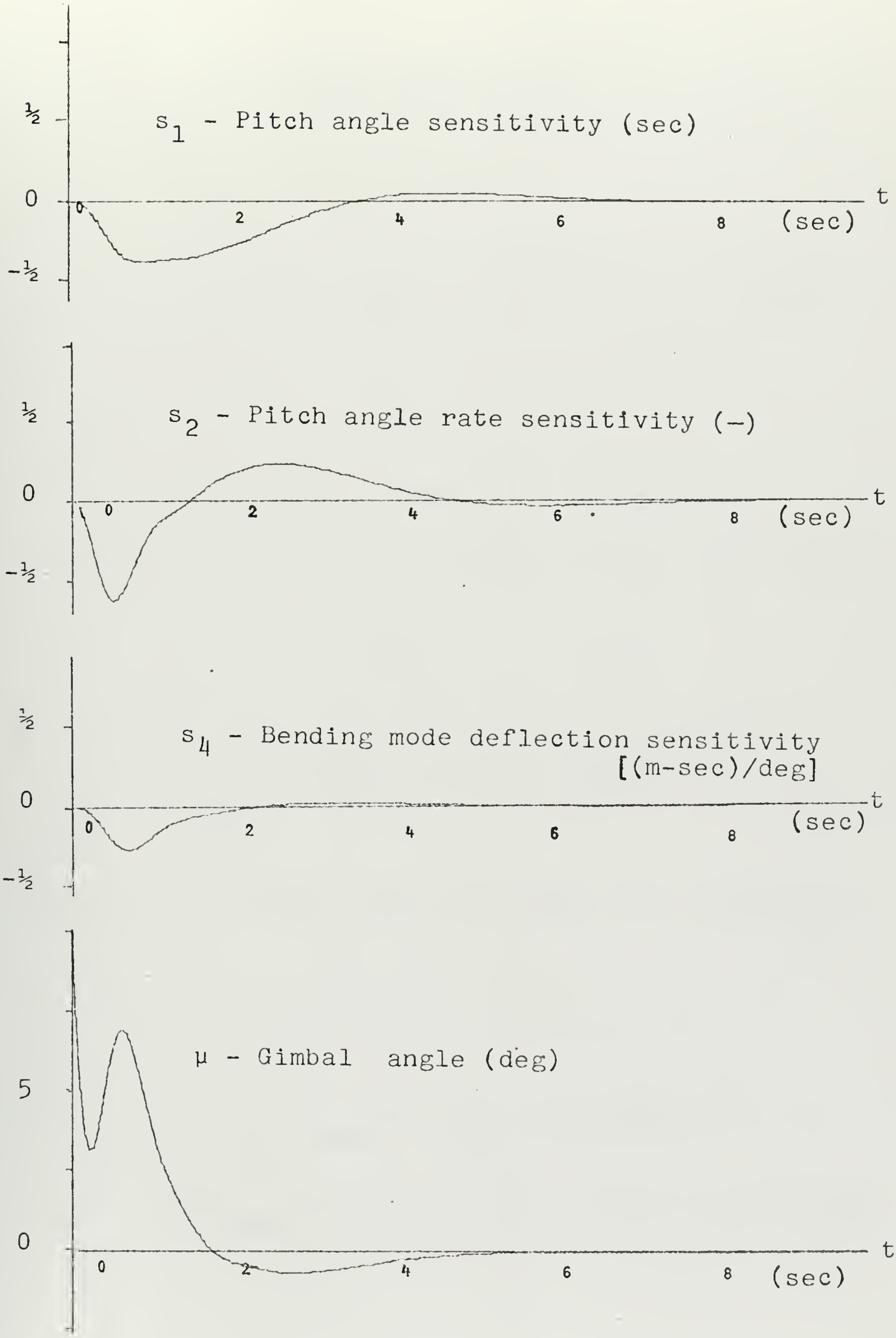

Figure 17. Cassidy and Lee Control: $\omega = \omega_o$; $Y = Y_o$.

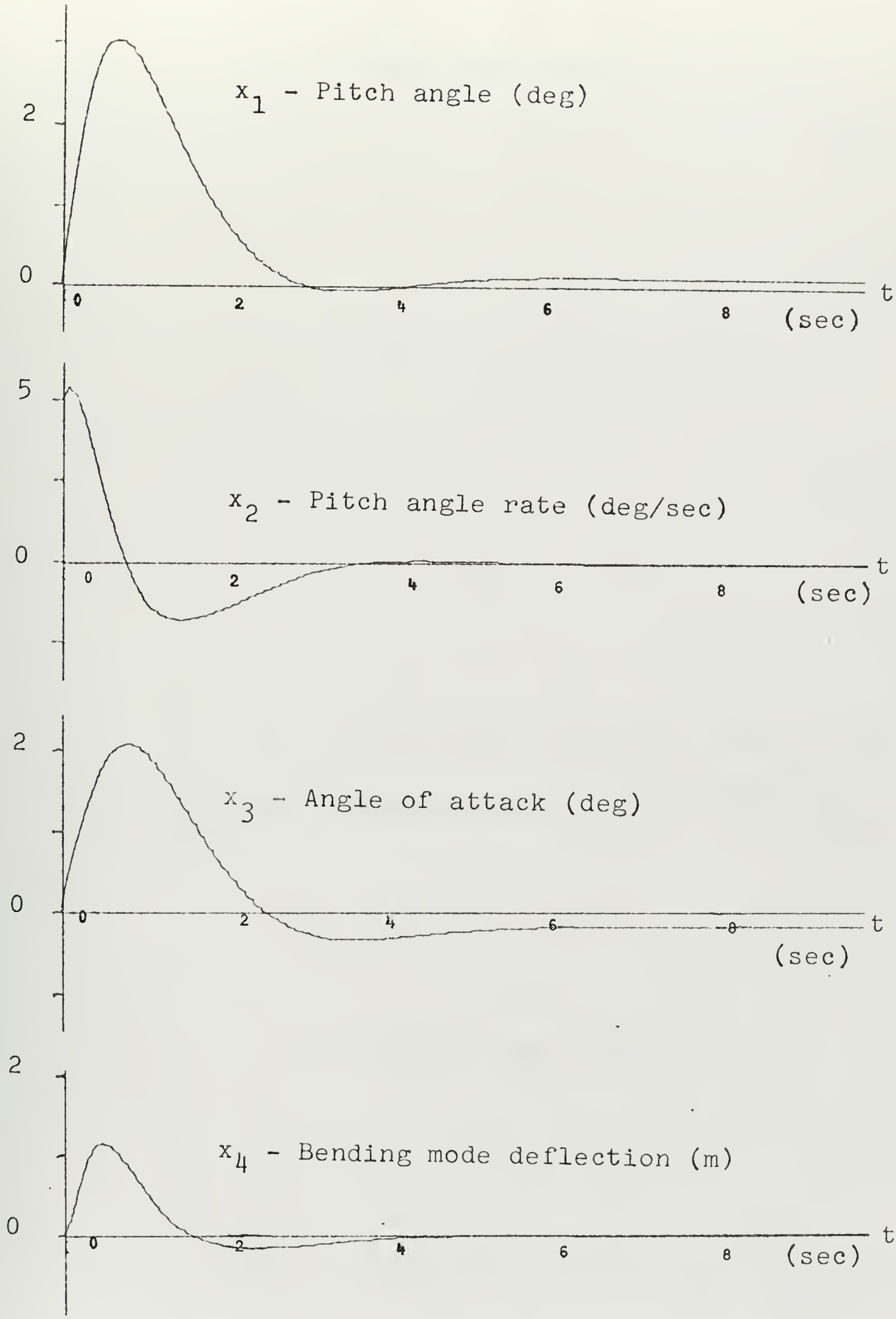

Figure 18. Cassidy and Lee Control: $\omega=0.8\omega_o$; $Y = Y_o$.

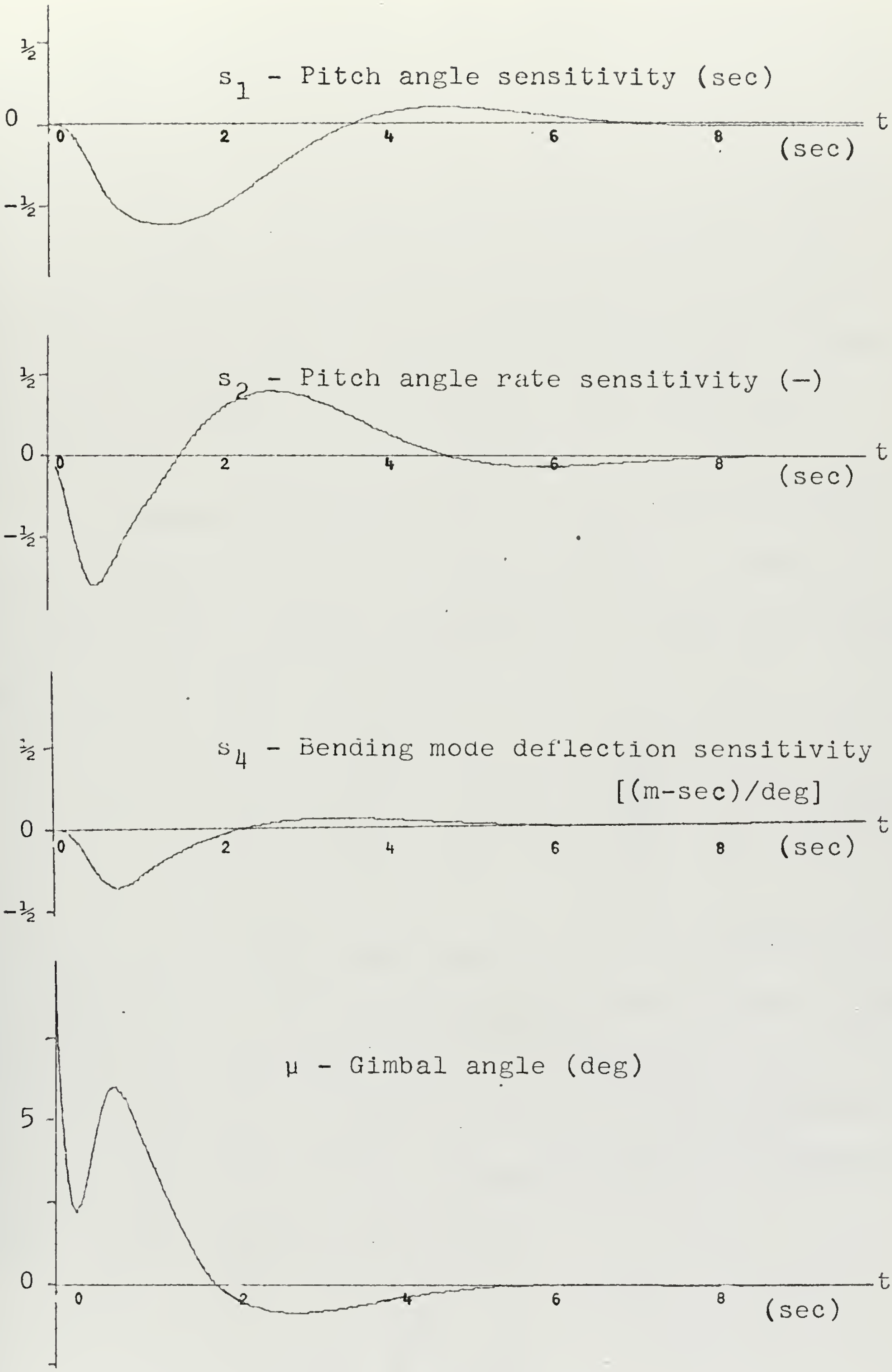

Figure 19. Cassidy and Lee Control: $\omega = \omega_o$; $Y = Y_o$.

The weighting matrix, $\underline{Q}$, was obtained by guessing a suitable form, solving the Riccati equation, and obtaining the resulting system trajectories. The values in $\underline{Q}$ were perturbed until a system that was sensitive to variations in ω was obtained. In addition to the sensitivity requirement, the maximum control magnitude was to be less than 10° and pitch angle, angle of attack and the maximum magnitude of bending mode deflection were to be small. These criteria were partially achieved.

The weighting matrix, $\underline{W}$, was selected from considerations of control, trajectory and sensitivity dispersions as indicated by the integral measures J_μ, $J_{\underline{x}}$, and $J_{\underline{s}}$. A parameter, λ, was defined such that

$$\underline{W} = \lambda \left\{ \begin{array}{ccccc} 0 & & & & \\ & 0 & & \underline{0} & \\ & & 0 & & \\ & \underline{0} & & 1 & \\ & & & & 1 \end{array} \right\} \tag{5.46}$$

The Kahne solution was obtained for various values of λ. The values of J_μ, $J_{\underline{x}}$, and $J_{\underline{s}}$ were obtained, for each system determined by the values of λ, by integrating (5.26), (5.27), and (5.28) with $\omega = \omega_o$ and $t_f = 20$ sec. The integrations were repeated for $\omega = 0.8\omega_o$. Two additional integral measures $J_{\underline{Q}\underline{x}}$ and $J_{\underline{W}\underline{s}}$ were defined by

$$J_{\underline{Q}\underline{x}} = \int_0^{t_f} \underline{x}'\underline{Q}\underline{x}dt \tag{5.47}$$

and

$$J_{\underline{Ws}} = \int_{0}^{t_f} \underline{s}'\underline{W}\underline{s}dt \quad . \tag{5.48}$$

The values of these integrals for $\omega = \omega_o$ and $\omega = 0.8\omega_o$ with t_f = 20 seconds were determined. Table II contains the values of $J_{\underline{x}}$, $J_{\underline{Qx}}$, $J_{\underline{s}}$, $J_{\underline{Ws}}$, and J_μ determined for the Kahne control system with $\omega = \omega_o$. Values were determined corresponding to a range of values for λ from $\lambda = 0$ (the optimal control) to $\lambda = 1$. The Cassidy and Lee solution for λ=.01 is also included. Table III contains the same information for $\omega = 0.8\omega_o$.

From the data in Tables II and III the trade-off curves of Figures 20 - 23 were obtained. Figure 20 shows the dispersion of $J_{\underline{x}}$ versus the dispersion of $J_{\underline{s}}$; this result is contrary to that obtained by Cassidy and Lee as shown in Figure 10 of [22]. The results indicated by Figure 20 were as expected for this problem, however, since the weighting matrix $\underline{Q}$ used in the performance measure indicated a lack of concern about states x_4 and x_5. It was expected that they would contribute heavily to $J_{\underline{x}}$ as they did for the optimal case. The weighting matrix $\underline{W}$ on the other hand was such that emphasis was placed on states x_4 and x_5.

Figures 21 and 22 indicate the cost in terms of control required to obtain a measured value of state and sensitivity trajectory dispersion respectively. Figure 23 indicates the optimal cost $J^* = J_{\underline{Qx}} + J_\mu$, as a function of sensitivity trajectory dispersion.

TABLE II

INTEGRAL MEASURE VALUES

$\omega = \omega_o$

λ	$J_{\underline{x}}$	$J_{\underline{Qx}}$	$J_{\underline{s}}$	$J_{\underline{Ws}}$	J_{μ}
0 [1]	41.0	3.4×10^{-3}	1,550	0.0	.0393
10^{-6}	20.1	3.4×10^{-3}	153	1.53×10^{-4}	.0401
10^{-5}	12.4	3.6×10^{-3}	32	3.16×10^{-4}	.0414
10^{-4}	7.4	4.1×10^{-3}	6.5	6.5×10^{-4}	.0438
10^{-3}	4.4	5.3×10^{-3}	1.5	1.5×10^{-3}	.0487
10^{-2}	2.6	8.1×10^{-3}	.42	4.2×10^{-3}	.0596
10^{-1}	1.5	1.7×10^{-2}	.15	1.5×10^{-2}	.0881
10^{-0}	.99	4.3×10^{-2}	.063	6.3×10^{-2}	.1176
C&L [2]	3.8	5.4×10^{-3}	.29	2.9×10^{-3}	.0491

[1] Optimal Solution

[2] Cassidy and Lee Solution with λ = .01.

TABLE III

INTEGRAL MEASURE VALUES

$\omega = 0.8\omega_o$

λ	$J_{\underline{x}}$	$J_{\underline{Qx}}$	$J_{\underline{s}}$	$J_{\underline{Ws}}$	J_{μ}
0	66.0	3.9×10^{-3}	146	0	.0392
10^{-6}	33.5	4.0×10^{-3}	80	8.0×10^{-5}	.040
10^{-5}	19.7	4.4×10^{-3}	30	2.9×10^{-4}	.0412
10^{-4}	11.5	5.4×10^{-3}	7.2	7.2×10^{-4}	.0436
10^{-3}	6.7	7.7×10^{-3}	1.8	1.8×10^{-3}	.0484
10^{-2}	3.8	1.4×10^{-2}	.54	5.4×10^{-3}	.0591
10^{-1}	2.1	3.4×10^{-2}	.20	2.0×10^{-2}	.0875

TABLE III (continued)

10^0	1.4	$1.1X10^{-1}$	.094	$9.4X10^{-2}$	.1774
C&L	5.7	$8.0X10^{-3}$	.476	$4.8X10^{-3}$	.0488

By studying the trade-offs indicated by Figures 21, 22, and 23, a choice of $\lambda = 10^{-2}$ or 10^{-3} becomes obvious. A value for $\lambda = 10^{-2}$ was chosen. Figure 20 was checked with this value and the choice was further confirmed.

This was the method used for choosing the weighting matrix $\underline{W}$; a trial-and-error method with added performance curves to aid in the design.

The Kahne solution for $\lambda = .01$ was chosen to be compared with the Cassidy and Lee solution for the same value of λ. This was done and the results indicated previously further confirmed the choice of λ.

E. USE OF $\underline{Q}$ MATRIX FOR REDUCED SENSITIVITY

Kalman [16] proved that the optimal feedback control law μ^* that minimizes the performance

$$J = \tfrac{1}{2}\int_0^{t_f} [\gamma \underline{x}'\underline{\underline{Q}}\underline{x} + \mu^2]dt \qquad (5.49)$$

and satisfies the constraints of a completely controllable linear single-input plant can be desensitized by the appropriate choice of γ. The assertion is that as γ increases the return difference increases and sensitivity to plant parameter decreases. The example of this chapter was solved for $\gamma = 2.0$ and for $\gamma = 10.0$.

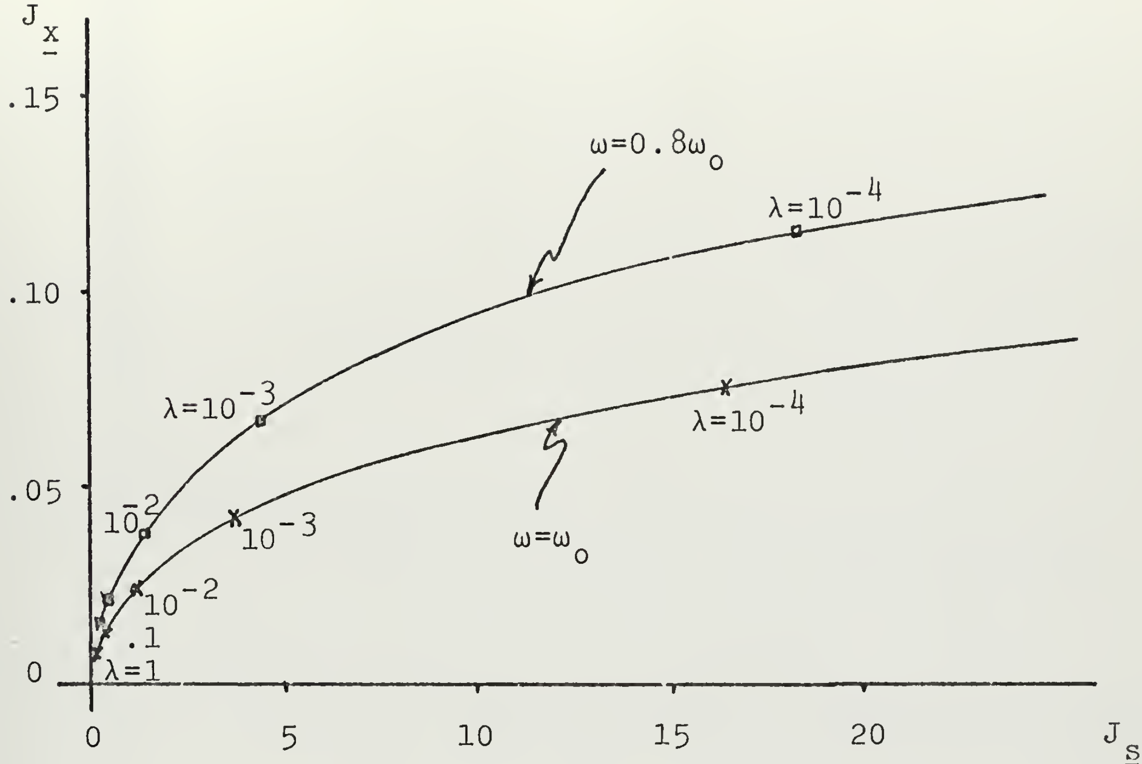

Figure 20. $J_{\underline{x}}$ vs. $J_{\underline{s}}$, trade-off data.

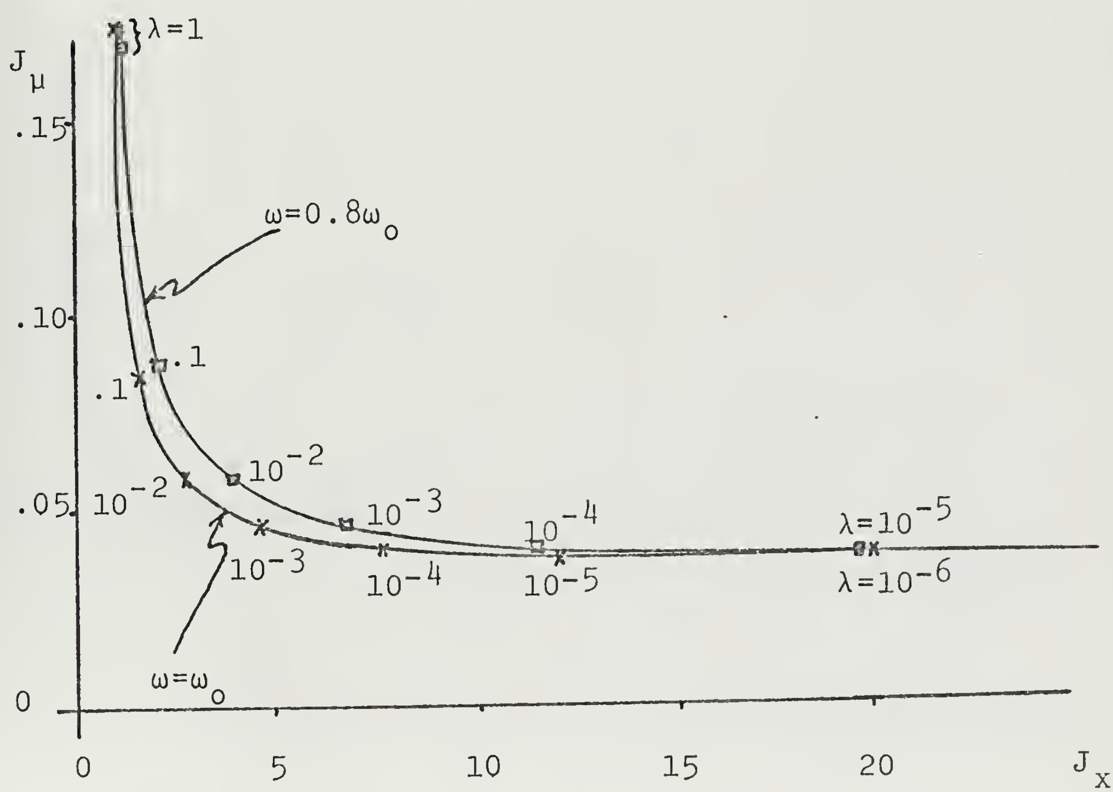

Figure 21. J_μ vs. $J_{\underline{x}}$, trade-off data.

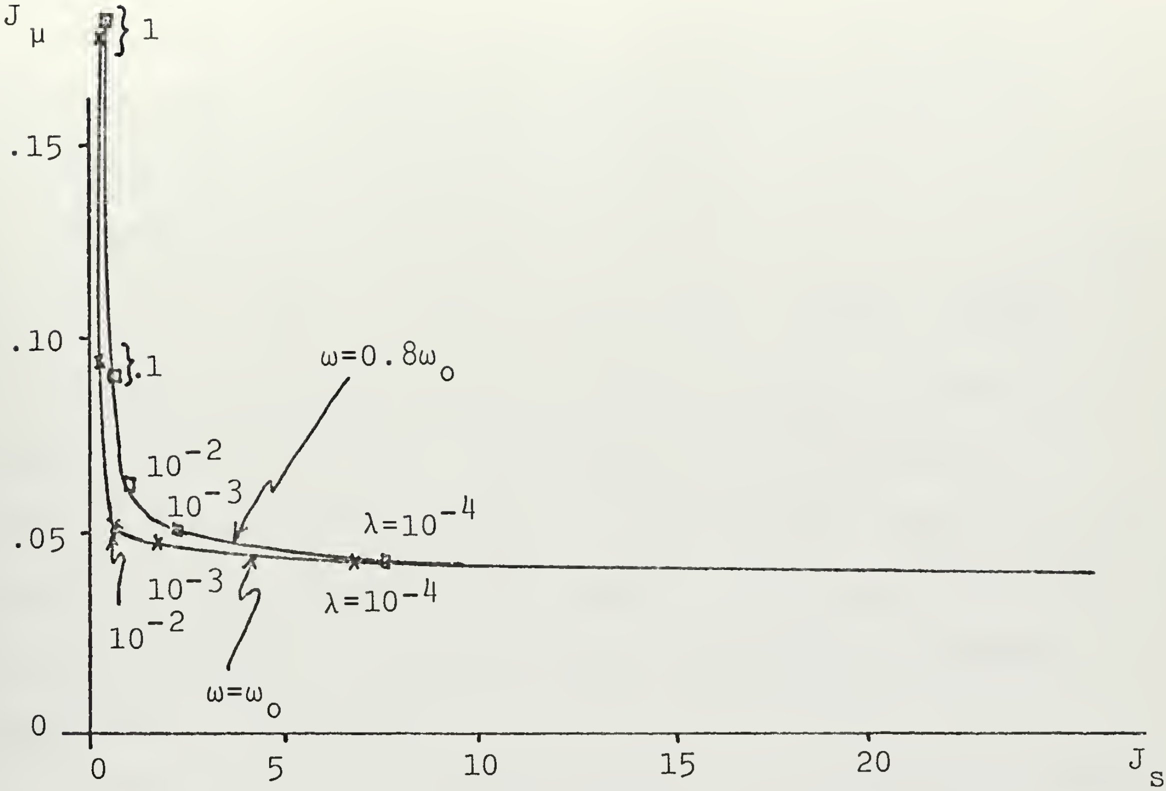

Figure 22. J_μ vs. $J_{\underline{s}}$, trade-off data.

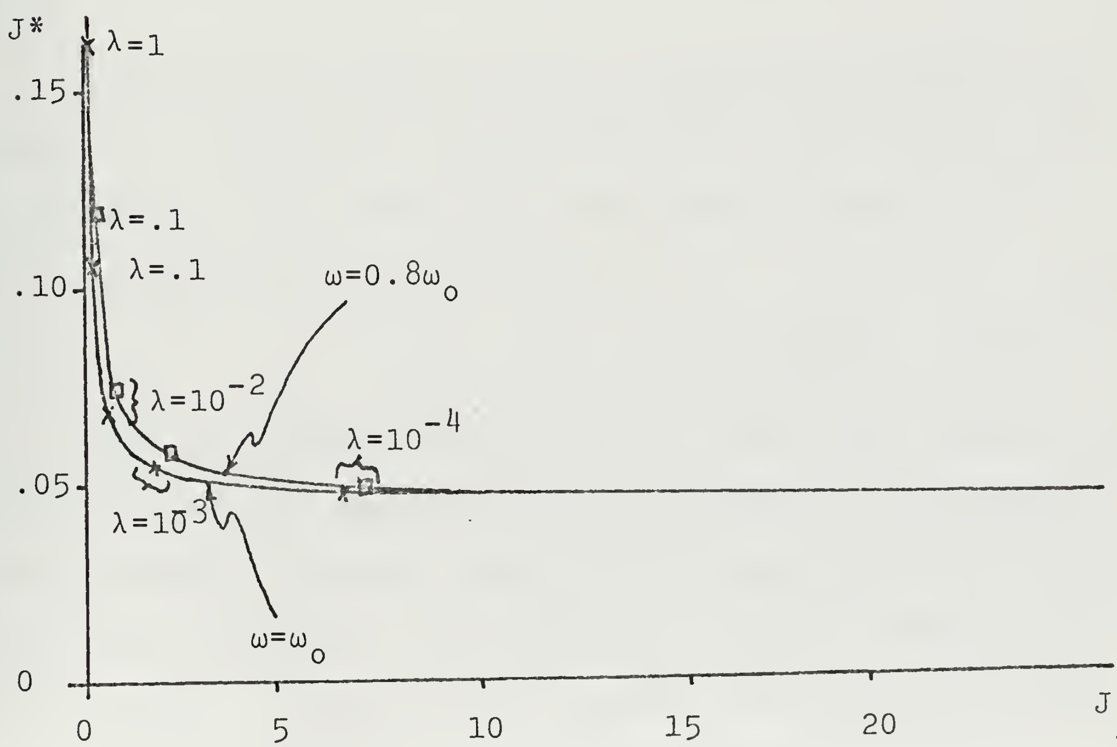

Figure 23. J* vs. $J_{\underline{s}}$, trade-off data.

The solutions were obtained in exactly the same way that the optimal solution was obtained except that the weighting matrix $\underline{Q}$ was multiplied by γ.

The gains obtained for $\gamma = 2.0$ were

$$\underline{f}' = \{ 1.77 \quad 2.36 \quad 1.29 \quad -.0346 \quad -.0186\}. \tag{5.50}$$

Figure 24 shows the trajectories x_1, x_2, x_3, and x_4 versus time for $\omega = \omega_o$. The trajectories were almost identical with those of Figure 9, the optimal case, $\gamma = 1.0$; the maximum value of x_4 was larger, 1.55 compared to 1.4. The trajectories of Figure 24 were slightly more damped than those of Figure 9.

Figure 25 shows trajectories x_1, x_2, x_3, and x_4 versus time for $\omega = 0.8\omega_o$. These trajectories were substantially the same as the comparable optimal trajectories of Figure 11.

Although the maximum control amplitude increased by about 1.2°, there was not a significant improvement in the sensitivity of the system, as indicated by Figure 26.

The gains for $\gamma = 10.0$ were

$$\underline{f}' = \{3.97 \quad 3.74 \quad 2.69 \quad -.0886 \quad -.0303\}. \tag{5.51}$$

With a maximum control value of 19°, twice that for the optimal solution, a system with considerably more damping was obtained. However, the maximum bending mode deflection was also increased. The results are shown in Figures 27, 28, and 29. As predicted by Kalman, the resulting system was less sensitive with $\gamma = 10.0$ than for $\gamma = 2.0$ or $\gamma = 1.0$

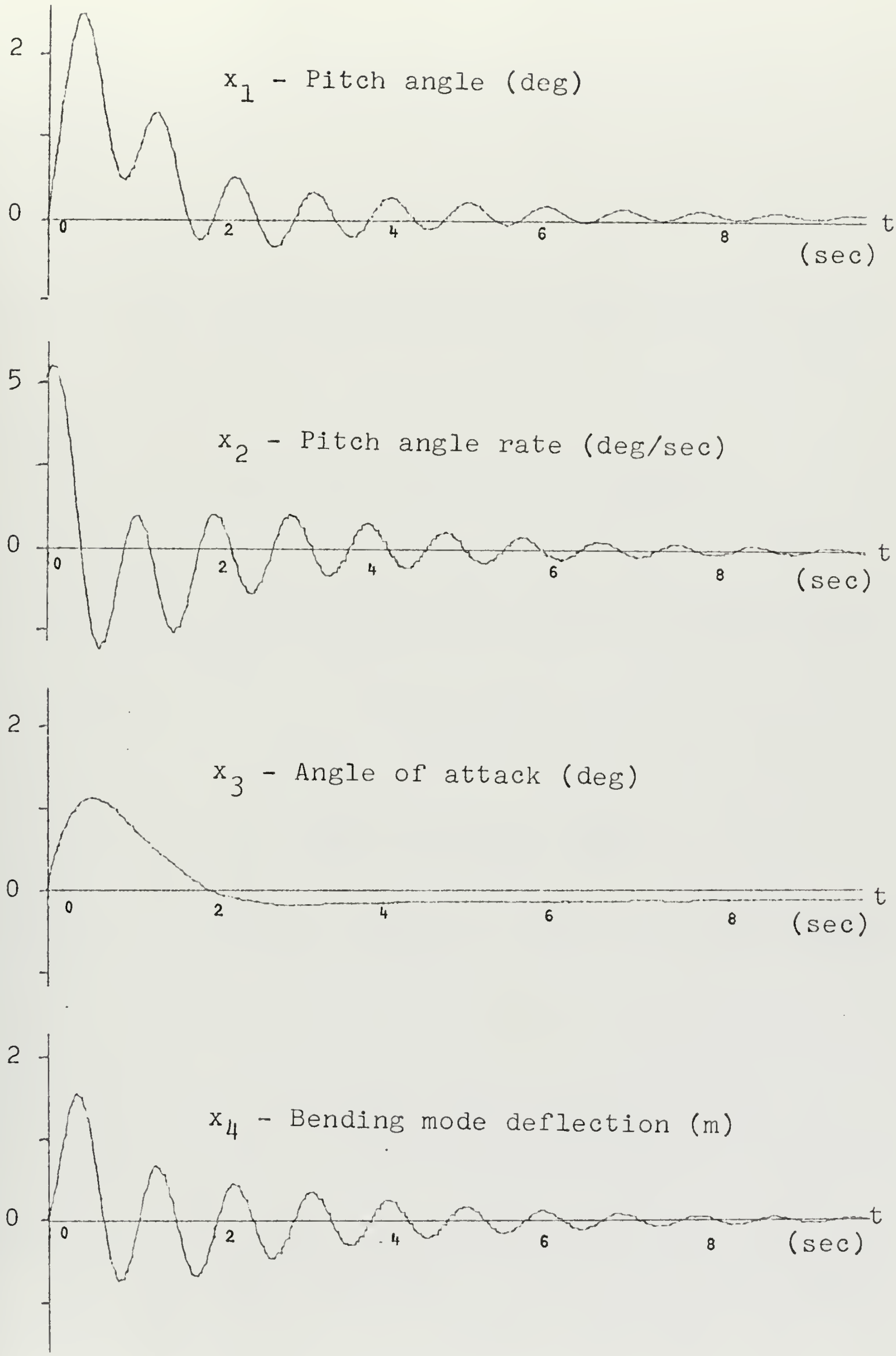

Figure 24. Optimal Control (γ=2); $\omega=\omega_o$; $Y=Y_o$.

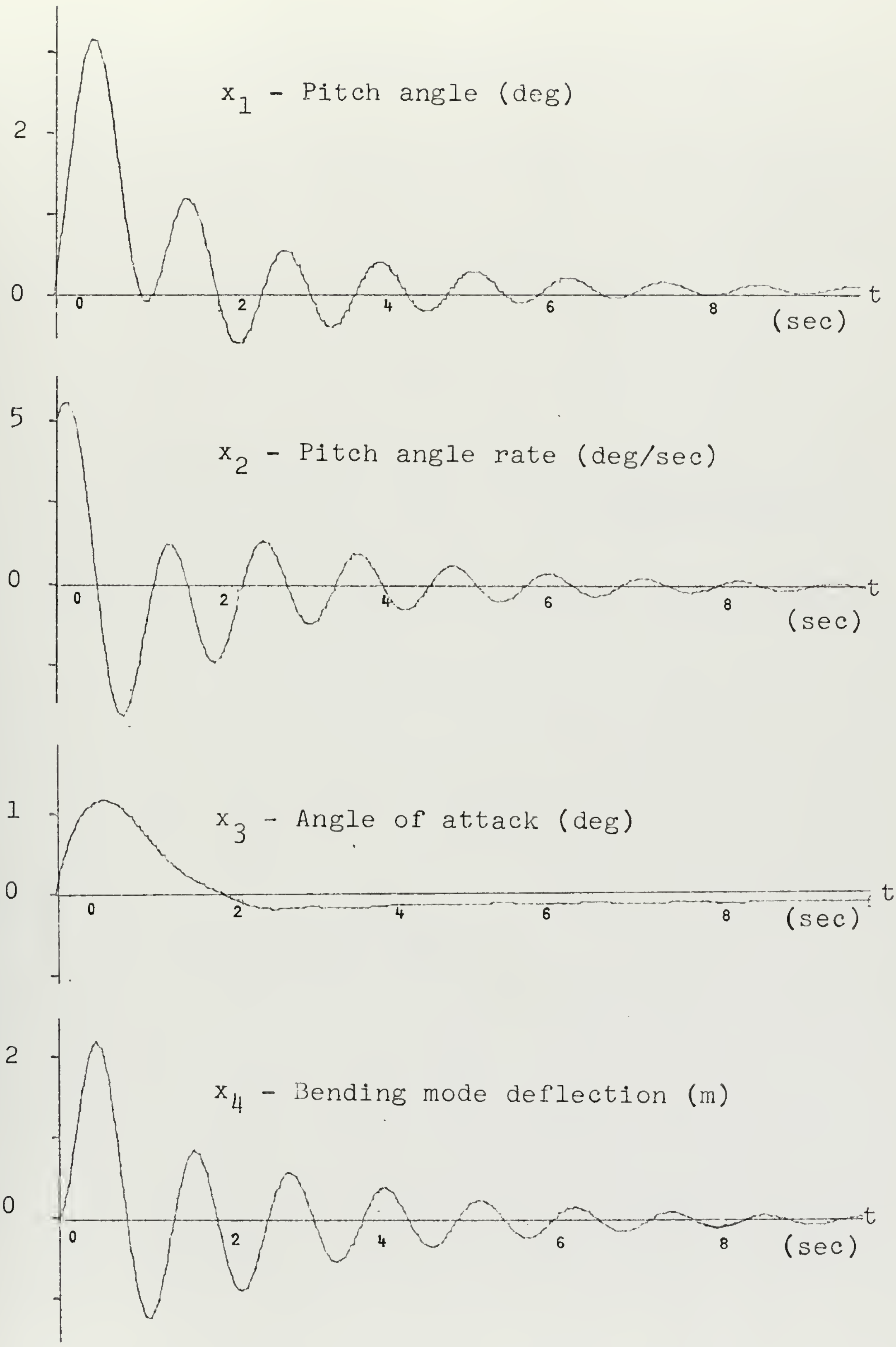

Figure 25. Optimal Control ($\gamma=2$): $\omega=0.8\omega_o$; $Y=Y_o$.

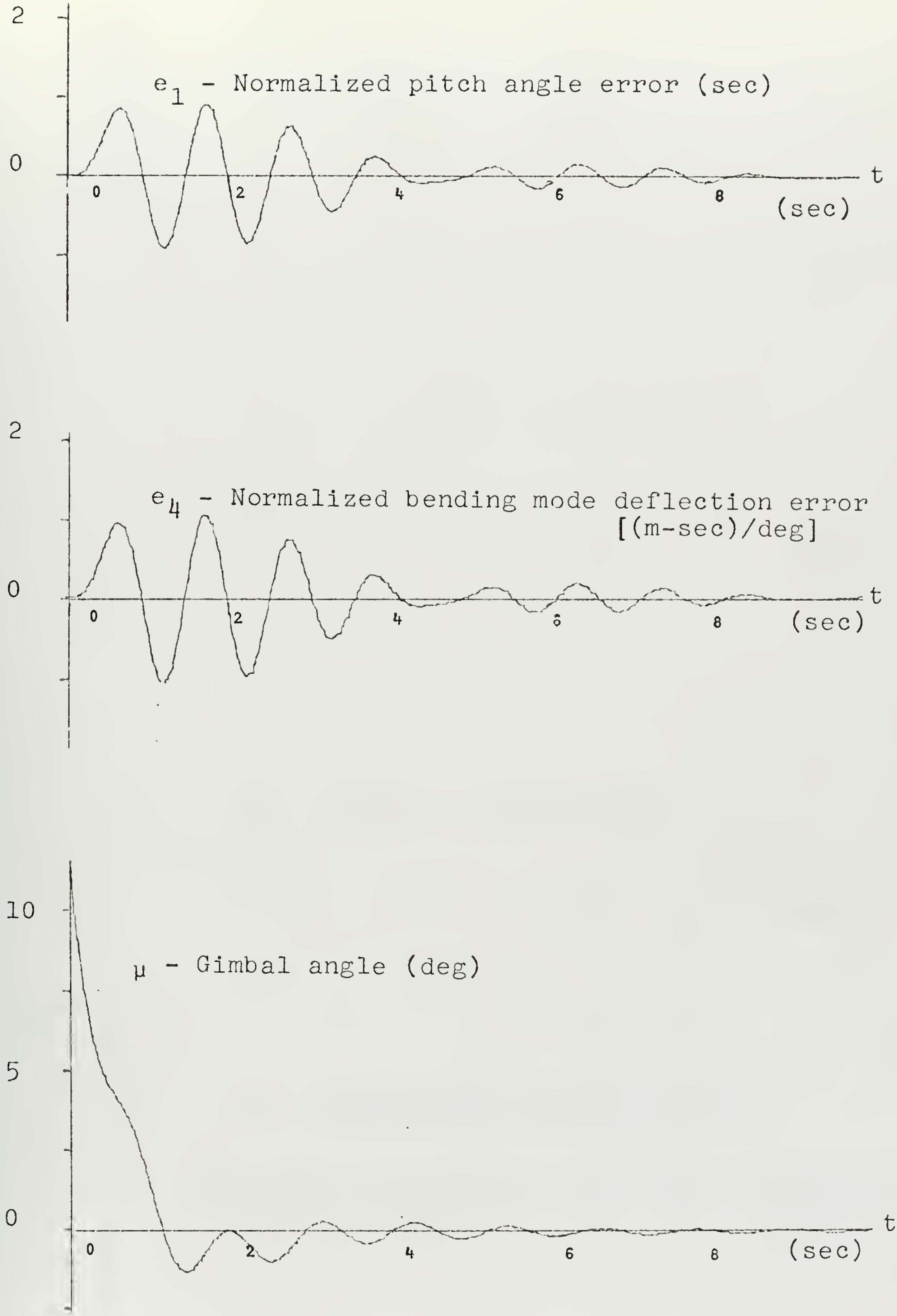

Figure 26. Optimal Control ($\gamma=2$): $\Delta\omega=\omega_o-0.8\omega_o$; $Y=Y_o$.

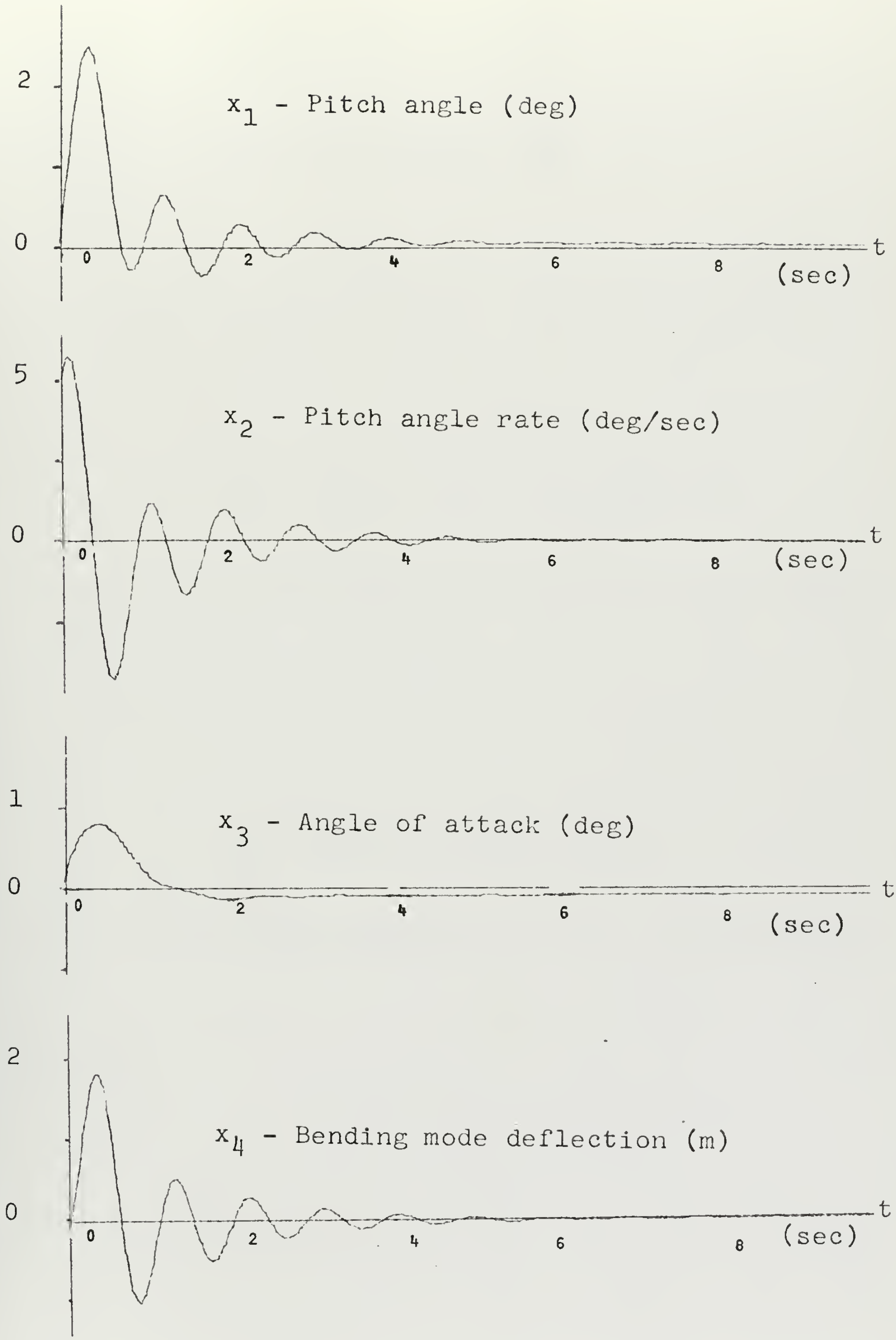

Figure 27. Optimal Control (γ=10): $\omega=\omega_o$; $Y=Y_o$.

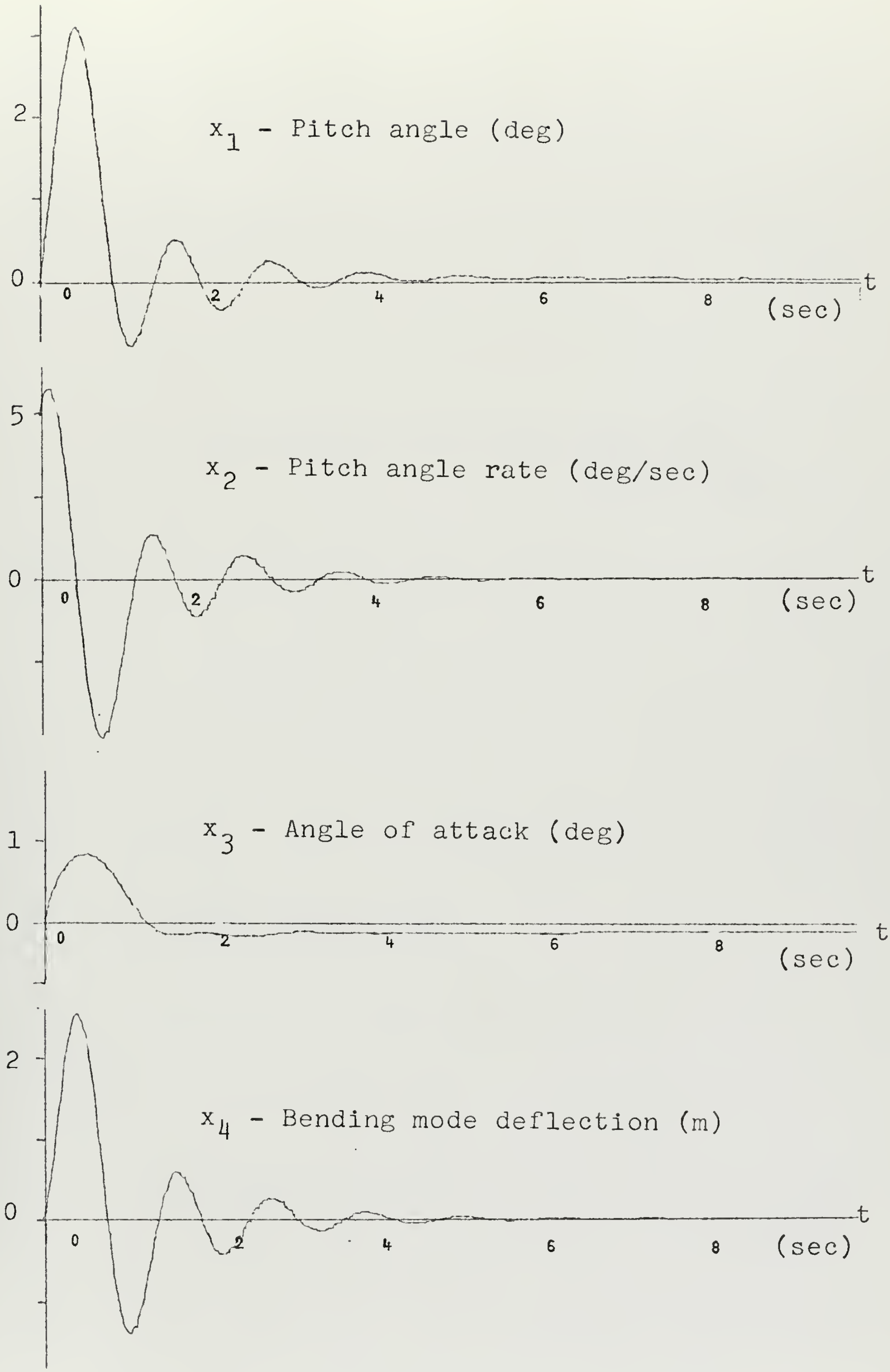

Figure 28. Optimal Control (γ=10): $\omega=0.8\omega_o$; $Y=Y_o$.

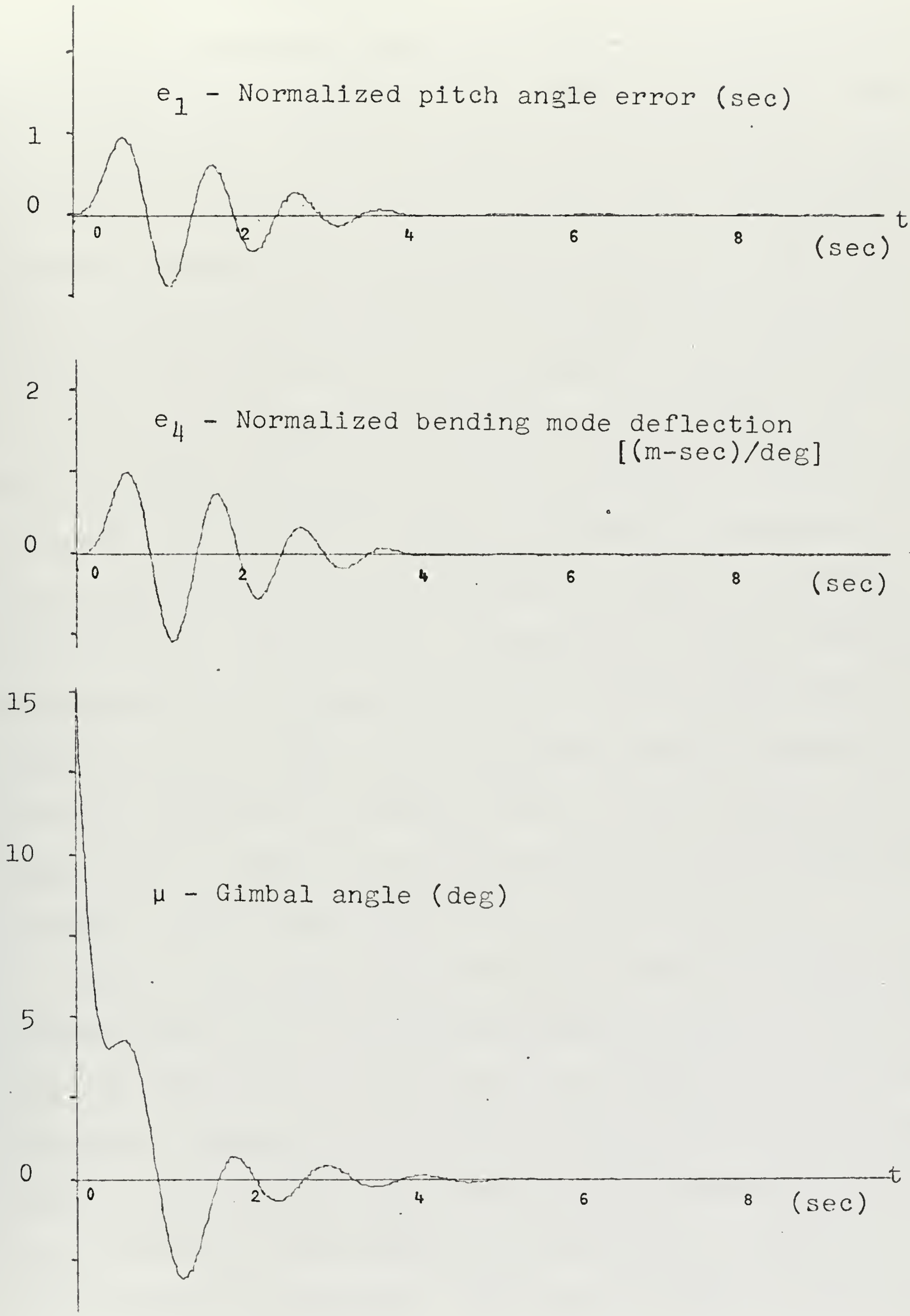

Figure 29. Optimal Control (γ=10): $\Delta\omega=\omega_o-.08\omega_o$; $Y=Y_o$

F. DISCUSSION OF RESULTS

All of the methods used resulted in a system with reduced sensitivity compared to the optimal solution. The Kahne and Cassidy and Lee solutions were obtained by considerably increasing the size of the problem (55 Riccati equations versus 15 for the optimal solution). The complexity of the problem increases enormously as additional parameters are considered. For example, if the parameters $\underline{q} = \{\omega \; Y'(x_r)\}'$ in the example had been used then 120 Riccati equations would have been integrated to obtain the solution. Application of these methods to a system in which the product $n(r+1)>10$ results in almost prohibitive computational difficulties. Integration of the 55 Riccati equations for the example with $t_f = 100$ sec. and using the largest possible time step that gave a stable solution, required 19 minutes of computer[2] time. This amount of computer time is quite large when compared with the two minutes required for the solution of the optimal control.

One of the advantages cited for using a technique that includes sensitivity in the performance measure is that it provides analytical control over the degree of insensitivity obtained. Figure 20 and Figure 21 show that this is true; however, one must add that obtaining the weighting matrix $\underline{W}$ that provides the degree of sensitivity required is a trial-and-error procedure. The data in Tables II and III

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[2] IBM - 360/67.

were obtained by solving the Kahne problem seven times and the Cassidy and Lee problem and the optimal problem each once. This required over 160 minutes of computer time, a very great cost for most designs. The point is that these techniques were trial-and-error techniques as was finding the optimal solution, because no method of determining the weighting matrices $\underline{Q}$ and $\underline{W}$ was available.

As stated earlier, there were two parameters to which the system was sensitive. The parameter $q = \omega$ was chosen for the example because it did not appear in the $\underline{b}$ matrix and therefore Kahne's technique was applicable. Desensitization of the system was achieved with respect to the parameter, ω, but what of the parameter, $Y'(x_r)$? There was nothing in the analysis technique that indicated how the "desensitized" system would respond to variations of a parameter not explicitly included in the performance measure. In fact it could very well happen that desensitizing with respect to one parameter might increase the sensitivity with respect to another. A designer having obtained a tentative solution to his control problem must check the sensitivity of the system to variations of the other plant parameters. In this example a check on sensitivity to $Y'(x_r)$ only was made.

The optimal control, the Kahne control, and the Cassidy and Lee control were each applied to the plant model with the parameters $Y'(x_r) = 1.2Y_o'(x_r)$ (increasing $Y'(x_r)$ produced the greatest sensitivity). The optimal system was

unstable in this case as indicated by trajectories x_1, x_2, x_3, and x_4 shown in Figure 30. The values of the integral measures for $Y = 1.2Y_o$ were

$$J_{\underline{x}} = 1794$$

$$J_{\underline{s}} = 52{,}350$$

and

$$J_\mu = .0429.$$

The Kahne system response with $Y = 1.2Y_o$ was very nearly identical to the nominal system as shown in Figure 31 and Figure 32. The values for the integral measures for $Y = 1.2Y_o$ were

$$J_{\underline{x}} = 2.79$$

$$J_{\underline{s}} = 0.443$$

and

$$J_\mu = 0.0596.$$

These values were very little different from those obtained for $\omega = \omega_o$ and $Y = Y_o$ shown in Table II. In fact the system was less sensitive to variations in Y than it was for variations in ω.

The Cassidy and Lee system response with $Y = 1.2Y_o$ was also close to the nominal system as shown in Figure 33 and Figure 34. The variation was only a little larger than that noted for the Kahne controller. The values for the integral measures for $Y = 1.2Y_o$ were

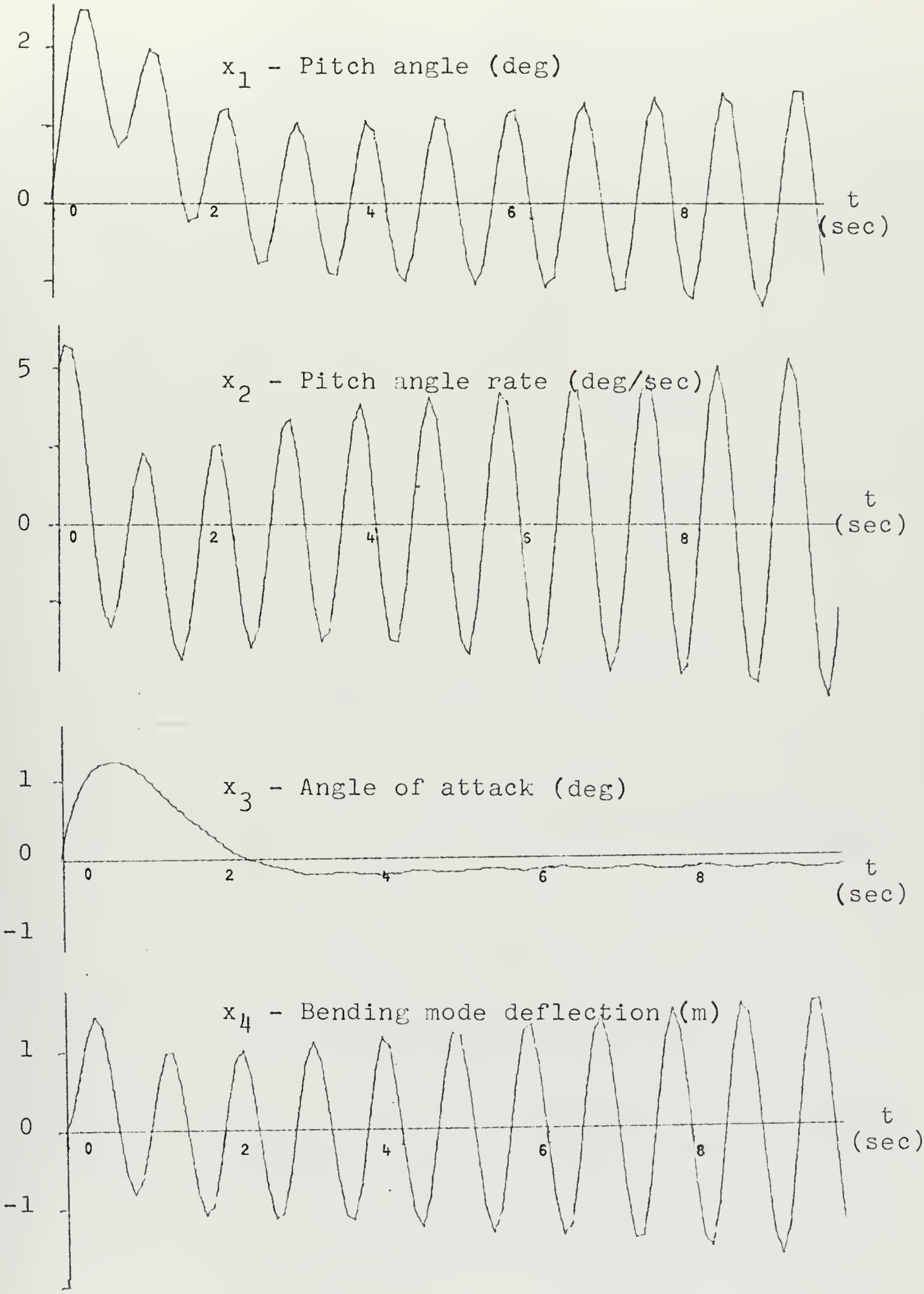

Figure 30. Optimal Control: $\omega = \omega_o$; $Y = 1.2Y_o$.

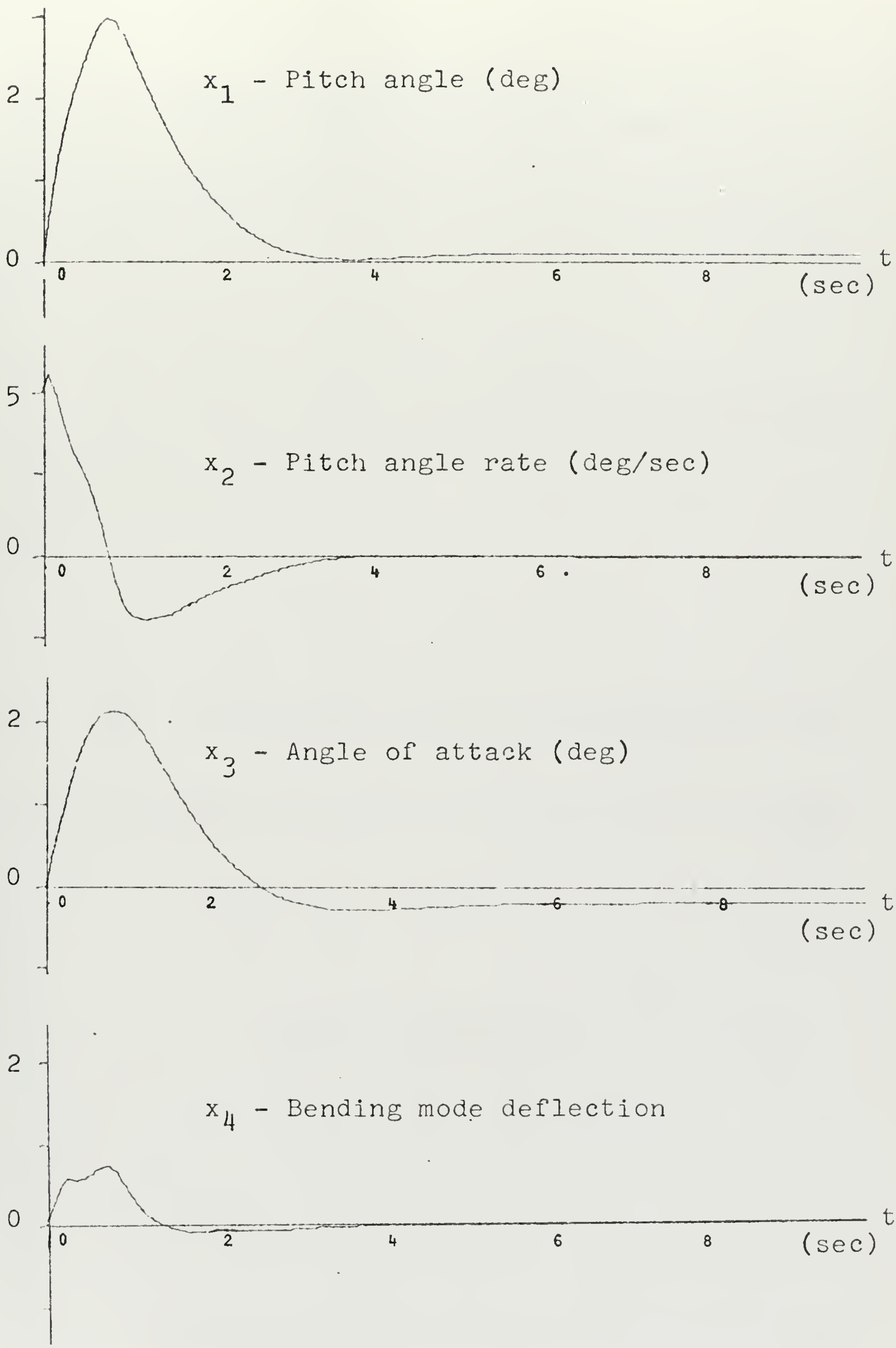

Figure 31. Kahne Control: $\omega = \omega_o$; $Y = 1.2Y_o$.

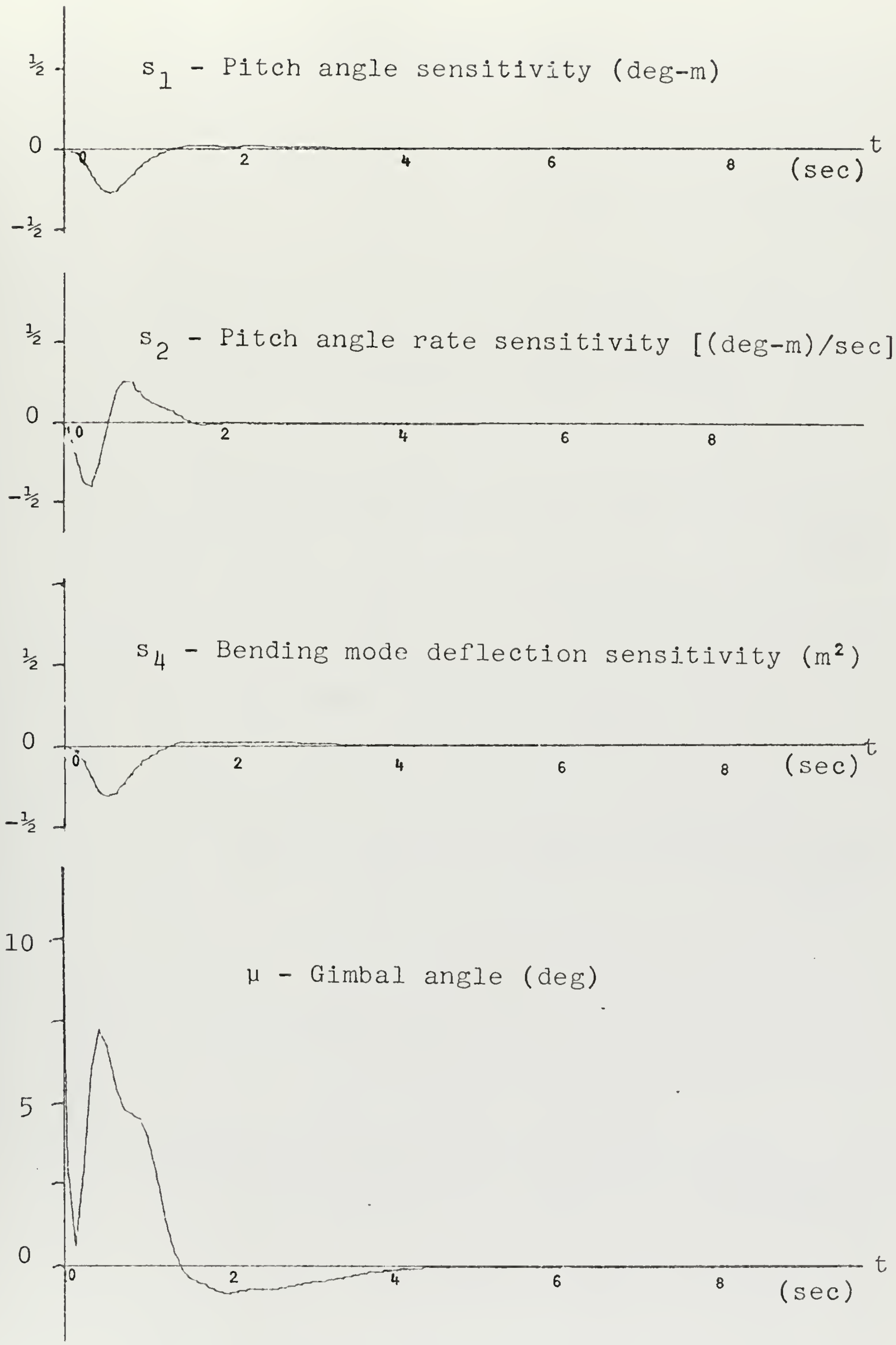

Figure 32. Kahne Control: $\omega = \omega_o$; $Y = 1.2Y_o$.

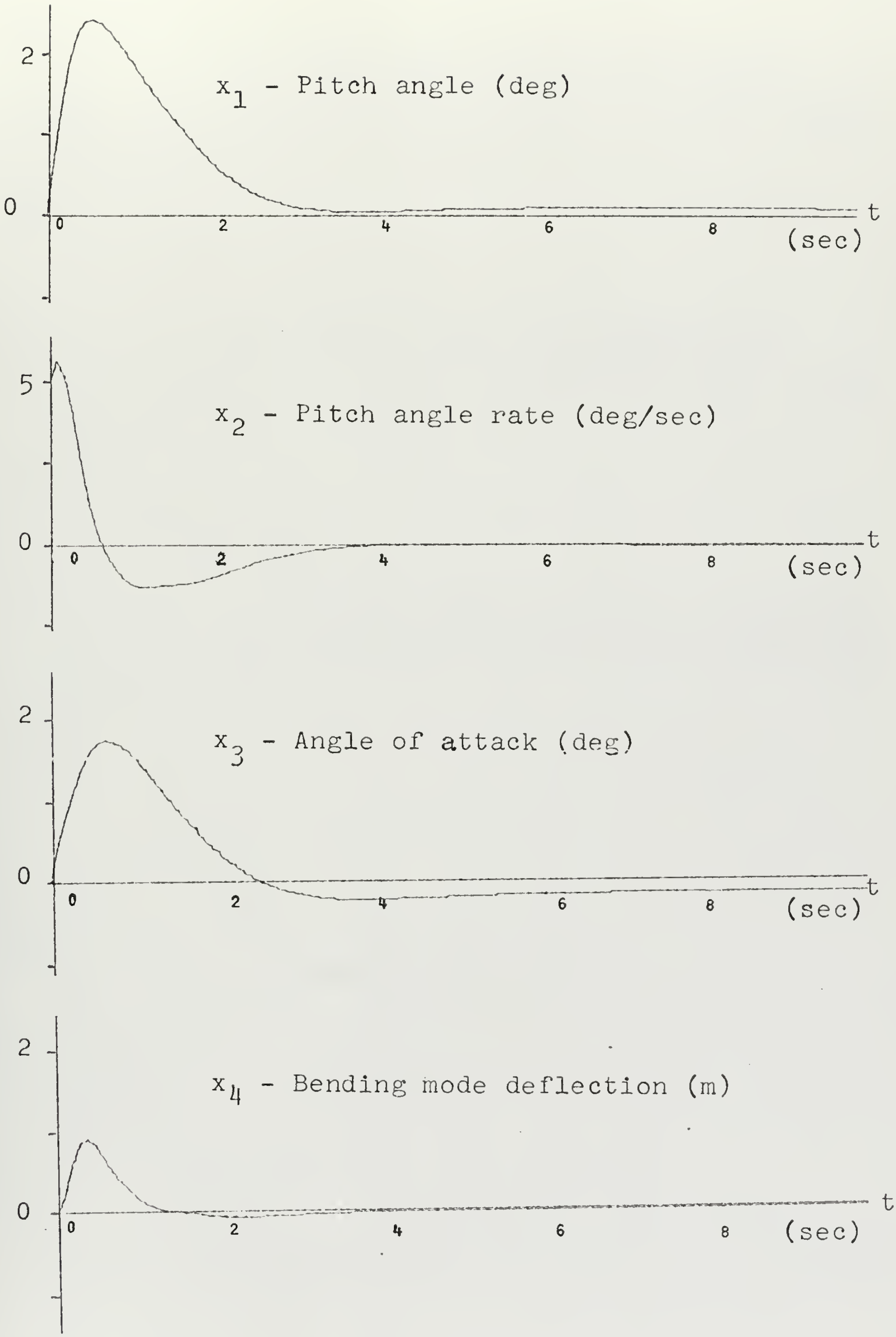

Figure 33. Cassidy and Lee Control: $\omega = \omega_o$; $Y = 1.2Y_o$.

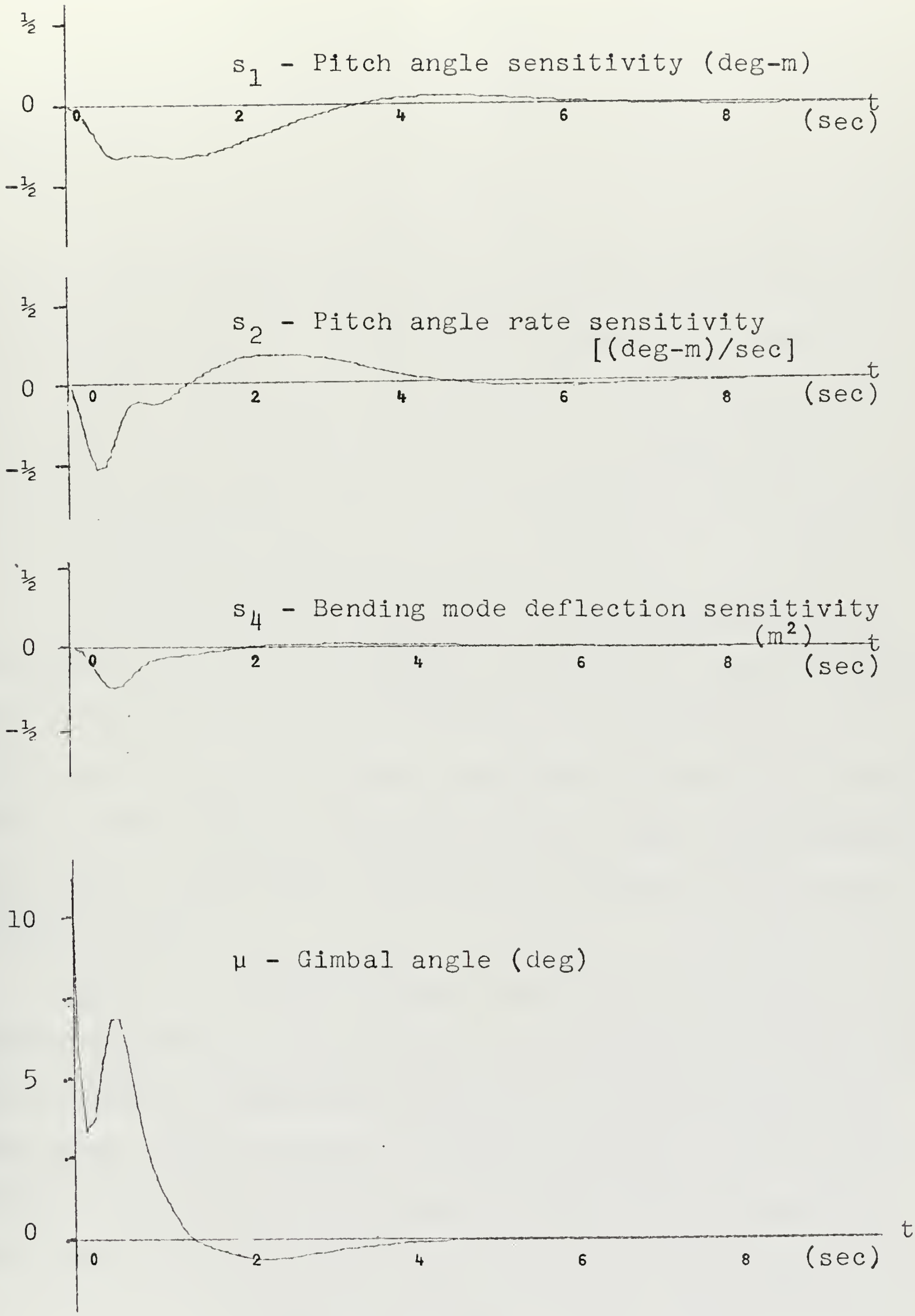

Figure 34. Cassidy and Lee Control $\omega=\omega_o$; $Y = 1.2Y_o$.

$$J_{\underline{x}} = 4.16$$

$$J_{\underline{s}} = 0.28$$

and

$$J_{\mu} = 0.049.$$

It is interesting to note that the system sensitivity with respect to ω for $Y = 1.2Y_o$ was slightly less than for the nominal case, $J_{\underline{s}} = 0.28$ compared to $J_{\underline{s}} = 0.29$. As was true for the Kahne controller, the Cassidy and Lee controller was less sensitive to variations in $Y'(x_r)$ than it was to variations in ω.

For this problem, the Kahne control system and the Cassidy and Lee control system provided very similar results in terms of sensitivity reduction as measured by the integral measure $J_{\underline{s}}$. The Kahne controller provided this result with a less complex controller and with less computational difficulty ($\underline{A}$ was not a function of elements of $\underline{K}$ for the Kahne solution).

Since the methods discussed were all essentially trial-and-error methods, perhaps more effort should have been put into finding a weighting matrix, $\underline{Q}$, for the optimal case that gave results similar to those of Cassidy and Lee and Kahne. It is not at all clear how one obtains weighting matrices that provide the desired result. For example, in the case of the problem presented in this chapter an attempt was made to solve the Riccati equation with $\underline{Q} = \underline{I}$. A solution could not be obtained because the integration quickly became numerically unstable.

A second optimal solution with weighting matrix

$$\underline{Q} = \begin{Bmatrix} .75 & & & & \\ & .01 & & \underline{0} & \\ & & 3.5 & & \\ & \underline{0} & & .01 & \\ & & & & .01 \end{Bmatrix} \tag{5.52}$$

was obtained. The feedback gains were

$$\underline{f}' = \{1.25 \quad 2.98 \quad 1.16 \quad -.246 \quad -.116\} \tag{5.53}$$

This control law resulted in a system with greatly improved sensitivity and performance, compared with the previous optimal solution, as indicated in Figures 35 - 38.

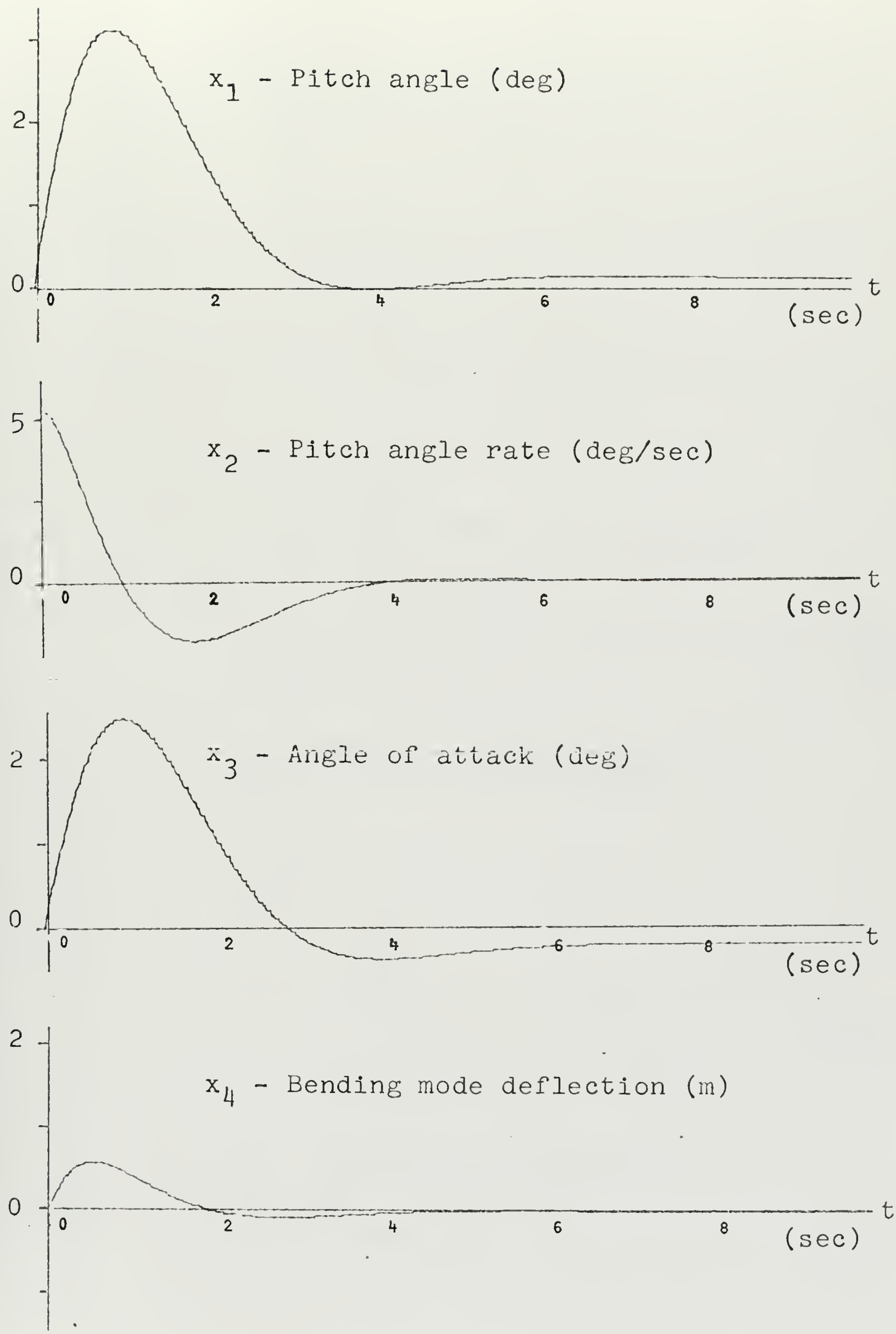

Figure 35. Optimal Control II: $\omega=\omega_o$; $Y=Y_o$.

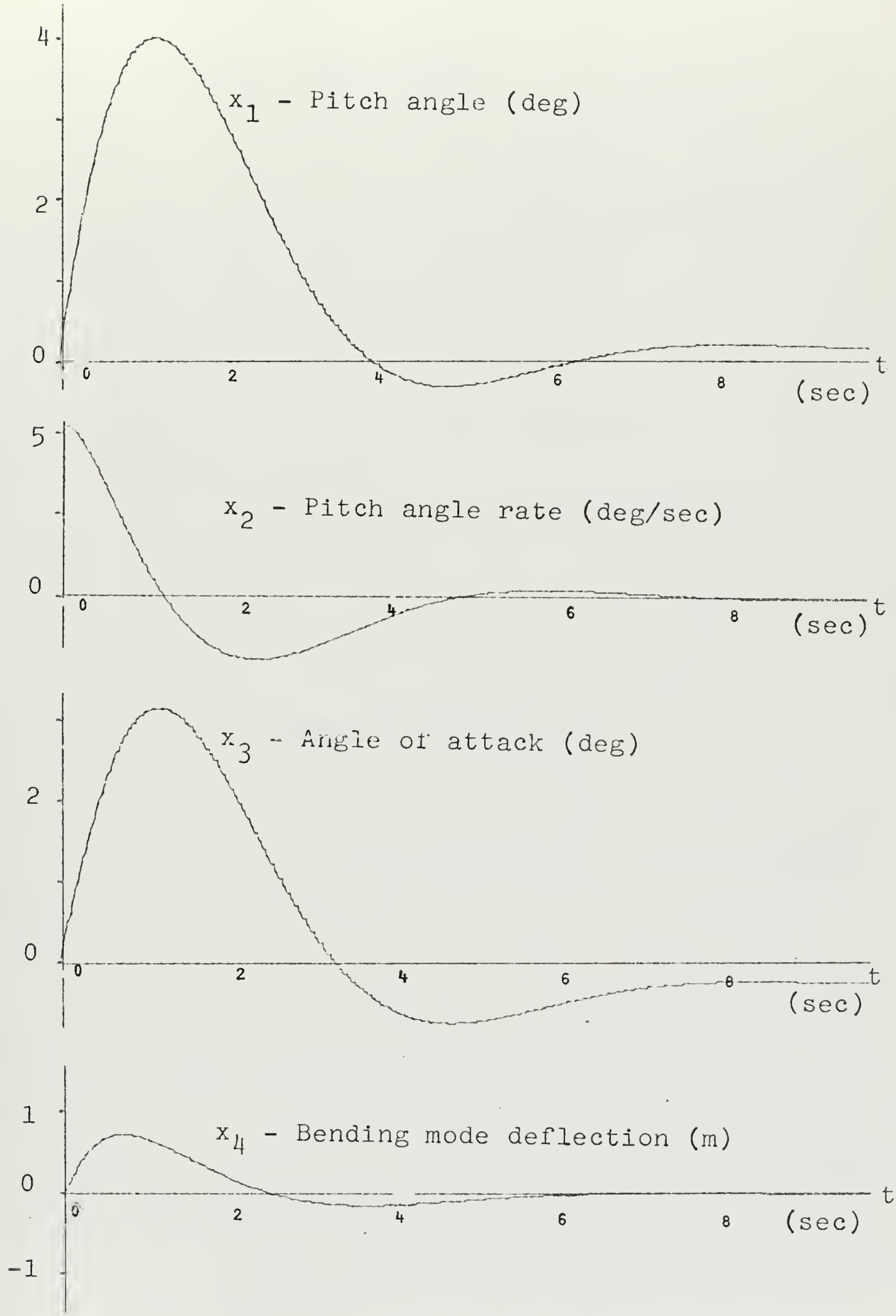

Figure 36. Optimal Control II: $\omega=0.8\omega_o$; $Y=Y_o$.

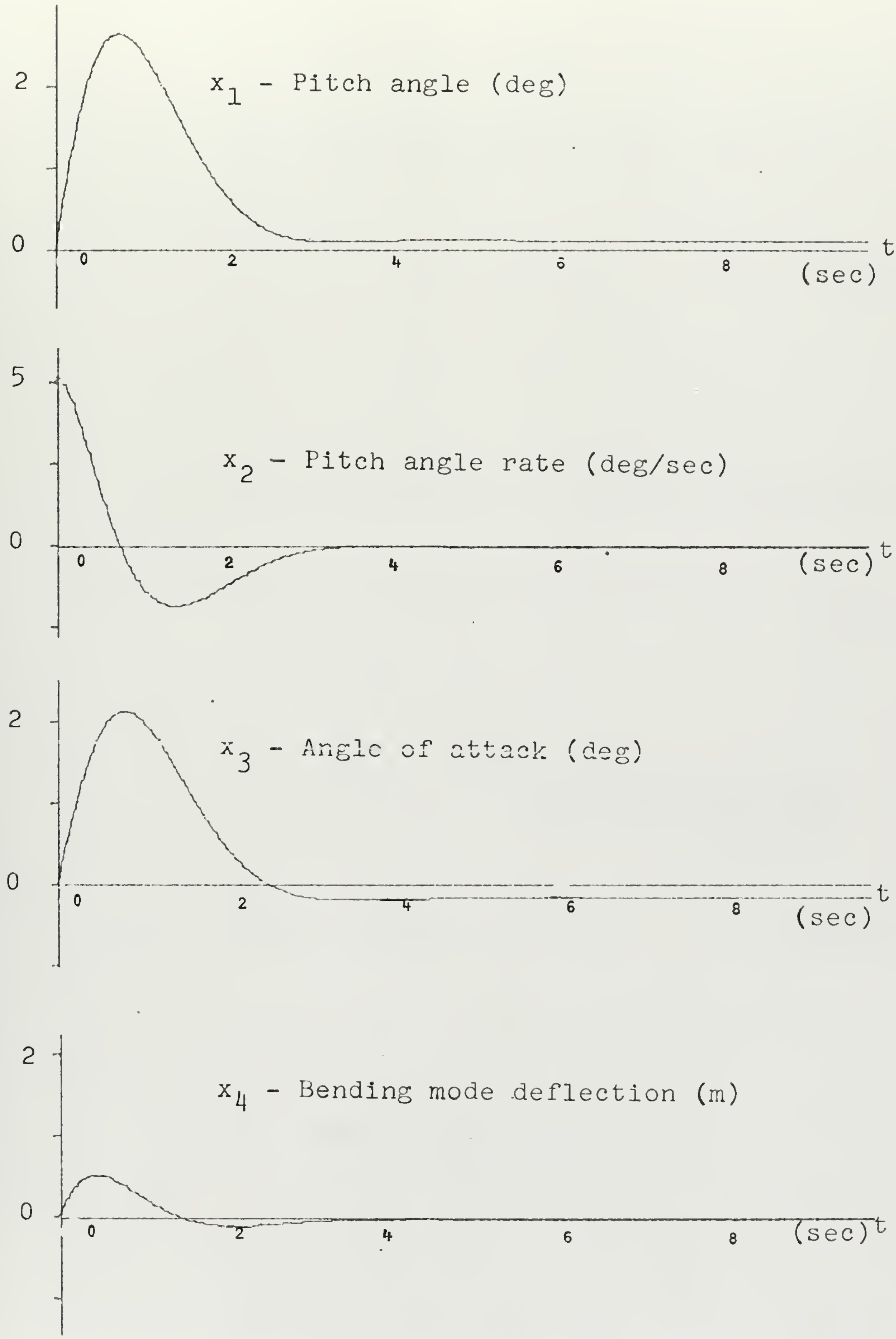

Figure 37. Optimal Control II: $\omega=\omega_o$; $Y=1.2Y_o$

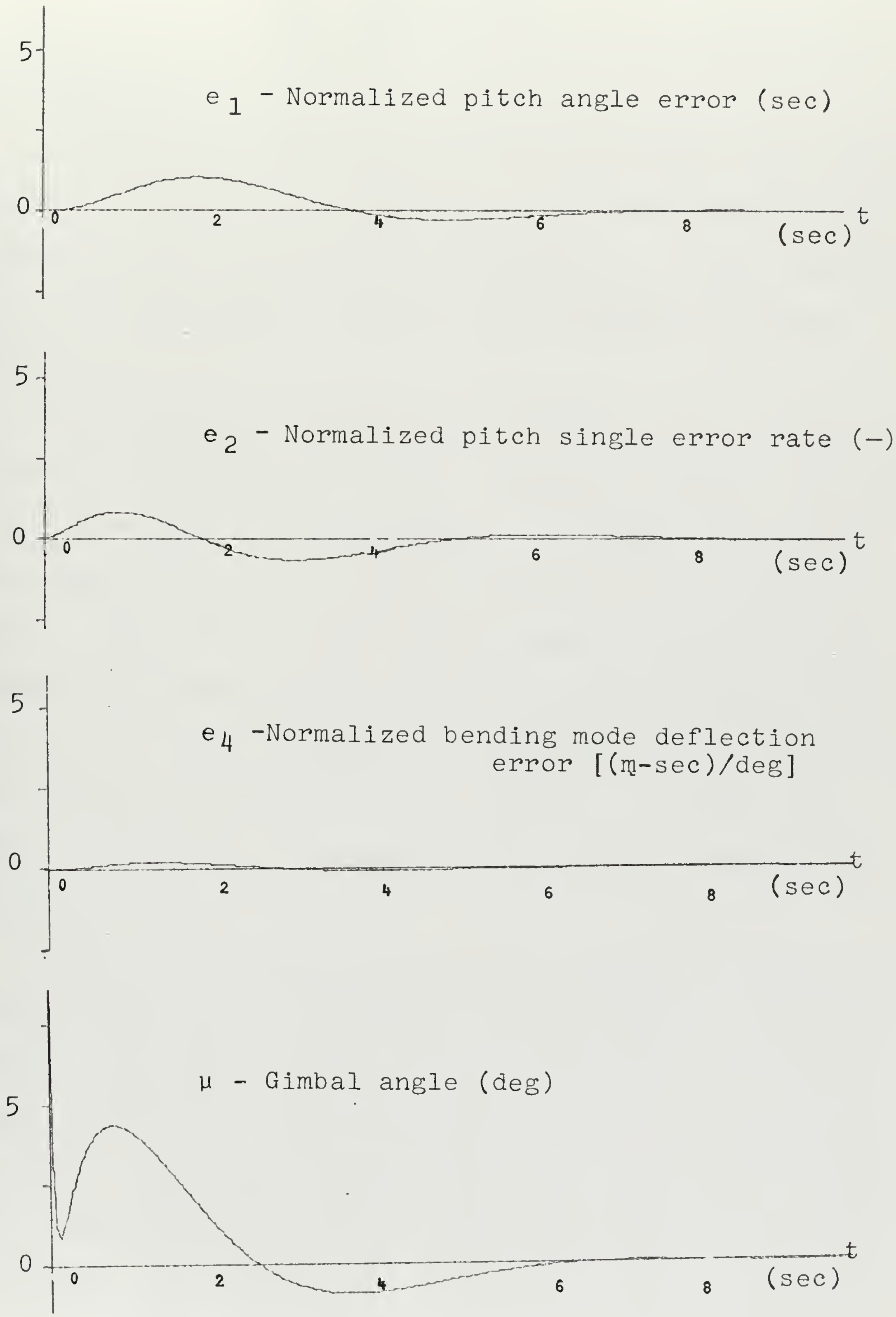

Figure 38. Optimal Control II: $\Delta\omega=\omega-0.8\omega_o$; $Y=Y_o$.

VI. CONCLUSIONS

A. SUMMARY

In this thesis several sensitivity analysis and design techniques as applied to optimal control systems have been evaluated. Of the methods of sensitivity analysis examined none provided a clear direction for extension to design techniques.

The basic results of Kalman, Perkins and Cruz, and Kreindler, applicable to linear regulator systems, provided an important bridge between optimal control theory in the time domain to classical control theory in the frequency domain. The direct relationship between stability and insensitivity demonstrated in the discussion of Kalman's work may contain a key to methods for applying some of the classical design techniques to the solution of desensitized linear regulators. This possibility will be discussed in the next section.

A sensitivity analysis model for optimal feedback systems was developed following well known methods. The resulting sensitivity equation was specialized to the linear regulator problem. The main disadvantage of this approach is that an independent analysis is required for each parameter considered. The Perkins and Wilkie analysis method eliminates this disadvantage for completely controllable linear regulators by solving for trajectory sensitivity to all parameters simultaneously. This is accomplished by transforming the state equations to the companion canonic

form, in which all of the parameters of the original system combine to form the coefficients of the open-loop characteristic equation.

Utilizing the sensitivity equation

$$\dot{\underline{s}}_i = \partial \underline{A}_i \underline{x} + \partial \underline{B}_i \underline{u} + \underline{A} \underline{s}_i + \underline{B} \frac{\partial \underline{u}}{\partial q_i} \tag{3.84}$$

an optimization problem with sensitivity constraints was formulated and solved for the linear case. The resulting control law, due to Higginbotham, proved to be very difficult if not impossible to implement. Two simplified techniques, resulting from simplifying assumptions and neglected terms in the sensitivity equation, as proposed by Cassidy and Lee, and Kahne were introduced. These schemes required augmenting the n^{th}-order system equation with r n^{th}-order sensitivity equations, and including a quadratic sensitivity term in the performance measure. The increase in size and computational difficulty of the resulting optimization problem severely limits the utility of these techniques.

In order to apply some of the techniques discussed, a 5^{th}-order linearized model of a flexible Saturn V-Apollo launch vehicle during booster powered flight was presented. An optimal (intentionally sensitive) solution to the resulting linear regulator problem was generated. The state equations were then augmented with the Kahne sensitivity equations and the resulting 10^{th}-order optimization problem was solved. The Cassidy and Lee solution was also obtained.

Both of these methods yield control laws that provide considerably reduced sensitivity compared to the optimal system.

Kalman suggested that sensitivity could be decreased by solving the modified optimization problem obtained by multiplying the trajectory weighting matrix, $\underline{Q}$, by a scalar γ. Solutions to the resulting problem with $\gamma = 2$ and $\gamma = 10$ were obtained. The resulting systems were indeed less sensitive than the optimal with $\gamma = 1$, but at a cost of considerably more control effort.

The problem was also solved as an unconstrained optimal control problem with a different trajectory weighting matrix, $\underline{Q}$. The resulting system had sensitivity characteristics similar to those obtained by the Cassidy and Lee, and Kahne methods.

With these results one is forced to ask "why bother with these extremely complex techniques, if the same result can be obtained by solving the much simpler unconstrained optimal control problem"? The answer is not at all clear since both approaches are trial-and-error with respect to determining weighting matrices. It was clear, however, that a better unconstrained design would be obtained if sensitivity were considered at each iterative step in the design process. Generating the data for the design curves of Figures 20-23, once the feedback gains were obtained, required much less computational effort than solving the Riccati equation. Similar design aids could be generated and used in obtaining the "best" unconstrained optimal solution.

B. RECOMMENDATIONS

A design technique for reduced sensitivity optimal controls which does not require augmenting the state equations or adding a term to the performance measure would be a highly desirable result. An area for further work with this in mind was found in Kalman's result for the inverse problem [16].

The basis for the design algorithm to be outlined is contained in the following ideas:

1) A completely controllable linear single-input system can be transformed into the companion canonic form.

2) The system characteristic equation is invariant under such a linear transformation.

3) A stable control law $\underline{f}$ for a completely controllable linear single-input system is an optimal control law if and only if

$$|1 - \underline{f}'\Phi(s)\underline{b}|^2 = |\Psi_k(s)/\Psi(s)|^2 > 1 \qquad (3.25)$$

or

$$|\psi_k(j\omega)|^2 - |\psi(j\omega)|^2 > 0. \qquad (6.1)$$

4) The polynomials ψ_k and ψ are the closed-loop and open-loop system characteristic polynomials.

5) For a system in companion canonic form, if the open-loop characteristic polynomial is

$$\Psi(s) = s^n + \alpha_n s^{n-1} + \ldots + \alpha_1 \qquad (6.2)$$

then the closed loop characteristic polynomial can be written as

$$\Psi_k(s) = s^n + (\alpha_n - \gamma_n)s^{n-1} + \ldots (\alpha_1 - \gamma_1) \quad (6.3)$$

where the γ_i are the elements of the feedback gain vector

$$\underline{f}' = \{\gamma_1 \ \gamma_2 \ \ldots \ \gamma_n\}. \quad (6.4)$$

6) The control law in the companion canonic form is given by

$$\mu = \underline{f}'\underline{z} \quad (6.5)$$

where $\underline{z}$ is the phase-variable state vector. Since

$$\underline{z} = \underline{T}\underline{x} \ , \quad (6.6)$$

because of the linear transformation to companion canonic form, the feedback gains in the original coordinate system are given by

$$\underline{k}' = \underline{f}'\underline{T} \ . \quad (6.7)$$

A design procedure that results from consideration of the listed ideas is the following:

1) Transform the completely controllable constant single-input system

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{b}\mu \quad (6.8)$$

into the companion canonic form using the Leverrier-Faddeeva algorithm [37]. This algorithm yields the characteristic polynomial, $\Psi(s)$ and the linear transformation, $\underline{T}$.

2) Obtain starting values for all γ_i in equation (6.4).

3) Using a search routine, find a set of γ_i, $i=1,2,\ldots n$, that make (6.3) Hurwitz. An eigenvalue finding routine and a scheme that reduces the real part of the largest eigenvalue would work well here. A design constraint on the location of

the largest eigenvalue can also be included here. The value of $(\alpha_1 - \gamma_1)$ must be constrained to be greater than α_1 as will be seen.

4) Having obtained a stable control law, $\underline{f}' = \{\gamma_1\ \gamma_2\ \ldots\ \gamma_n\}$, test condition (6.1) as follows:

a) $|\psi_k(j\omega)|^2 = \hat{\psi}_k(\omega^2)$

and similarly

$$|\psi(j\omega)|^2 = \hat{\psi}(\omega^2).$$

Therefore, condition (6.1) is reduced to

$$\hat{\psi}_k(\omega^2) - \hat{\psi}(\omega^2) > 0. \qquad (6.9)$$

b) Since $\hat{\psi}_k(\omega^2)$ and $\hat{\psi}(\omega^2)$ are nonnegative for all ω, and since in step 3 the constraint

$$\hat{\psi}_k(0) > \hat{\psi}(0) \qquad (6.10)$$

was imposed by requiring the condition $(\alpha_1 - \gamma_1) > \alpha_1$, then condition (6.9) holds for all ω if and only if the equation

$$\hat{\psi}_k(\omega^2) - \hat{\psi}(\omega^2) = 0 \qquad (6.11)$$

has no real roots.

c) The test then is to find the roots of (6.11), if none are real, then (6.9) is satisfied and $\underline{f}'$ is an optimal control law. If (6.11) has real roots, the process must be repeated until a suitable $\underline{f}'$ is found.

5) Having found $\underline{f}'$, obtain $\underline{k}'$ from

$$\underline{k}' = \underline{f}'\underline{T}. \qquad (6.7)$$

A variation to the procedure which limits the minimum value of the return difference is obtained by modifying condition (6.9) such that

$$\hat{\psi}_k(\omega^2) - \rho^2\hat{\psi}(\omega^2) > 0 \tag{6.12}$$

where

$$\rho > 1.$$

Such a procedure will ensure that the return difference is greater than ρ for all ω. That is condition (3.25) becomes

$$|1- \underline{f}'\underline{\Phi}(s)\underline{b}|^2 = |\Psi_k(s)/\Psi(s)|^2 > \rho^2 \tag{6.13}$$

There is a connection between the magnitude of ρ and the system sensitivity but it is not clear at this point. The choice of ρ, therefore, remains a problem to be solved.

This procedure, which could incorporate classical techniques for determining constraints for the eigenvalues of the closed-loop system, provides a simple bridge between optimal control systems and classical control systems. If, for example, consideration is limited to a range of frequencies, $\omega < \omega_1$, the control resulting would not be optimal but would satisfy the closed-loop requirements of a classical system.

It is very likely that this technique can be extended to multiple-input completely controllable systems using the arguments of Wilkie and Perkins [15]. Since the systems are linear, it follows from superposition that

$$\underline{x} = \underline{x}^1 + \underline{x}^2 + \ldots + \underline{x}^m \tag{6.14}$$

where $\underline{x}^i$ is the response to control μ_i, i = 1, ... m. The control laws for the decoupled single-input systems could be determined and recombined to form an optimal nXm gain matrix, $\underline{F}$. The nXm matrix $\underline{F}$ would have the form

$$\underline{F} = \{ \underline{k}_1 \quad \underline{k}_2 \quad \ldots \quad \underline{k}_m \} \tag{6.15}$$

where each $\underline{k}_i$ is determined from

$$\underline{k}_i = \underline{f}_i \underline{T}_i . \tag{6.16}$$

The $\underline{f}_i$ of (6.16) are obtained using the above procedure.

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13. ABSTRACT

A cohesive and detailed treatment of the theory and engineering implications of trajectory sensitivity is presented. Fundamental results that provide insight into the theoretical aspects of trajectory sensitivity analysis, in both the frequency and time domains, are reviewed. Several related methods for incorporating sensitivity considerations in the design of systems are presented and used to solve a meaningful fifth-order numerical example: a flexible space vehicle in booster powered flight. Comparisons are made between an optimal design and designs which include a sensitivity term in the performance measure and conclusions are drawn about the efficacy of these techniques in control system design.

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